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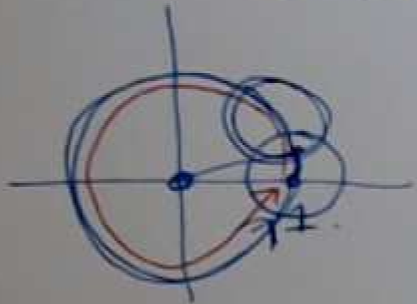
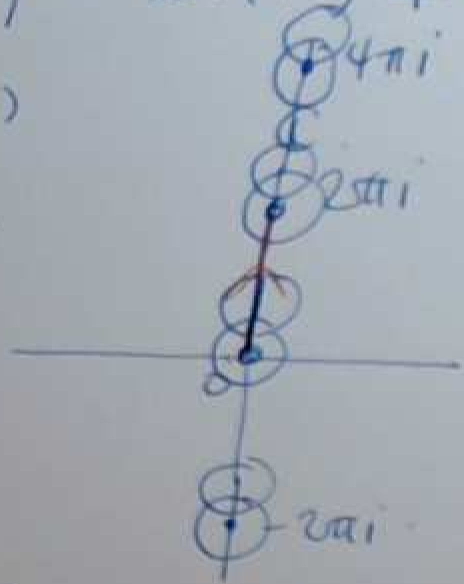
$\ln(z)$. $z = re^{i\theta}$.

$\ln(re^{i\theta}) = \ln(r) + i\theta + 2\pi i n$.

$\mathbb{C} \xrightarrow{\ln(z)} \;\xleftarrow{e^z}$





start w/ $\ln(1) = 0$. end up at $2\pi i$.
let $\ln(z)$ vary continuously

$i\theta$. $\log(i\theta) = \theta i + 2\pi i n$.

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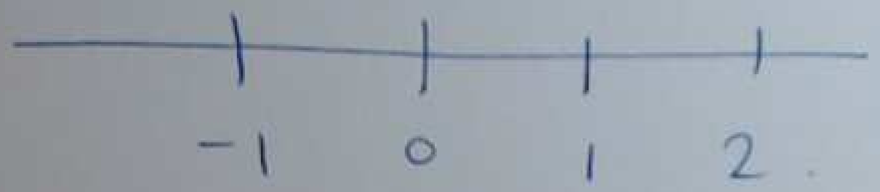
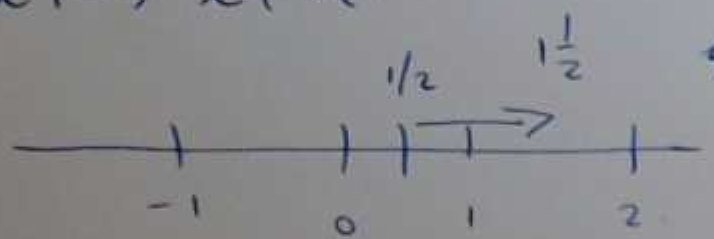
$\mathbb{R}$.



$\mathrm{Isom}^+(\mathbb{R}) \leftarrow$ orientation preserving isometries of $\mathbb{R}$.

$x \mapsto x + a$.



eg $x \mapsto x + 1$.

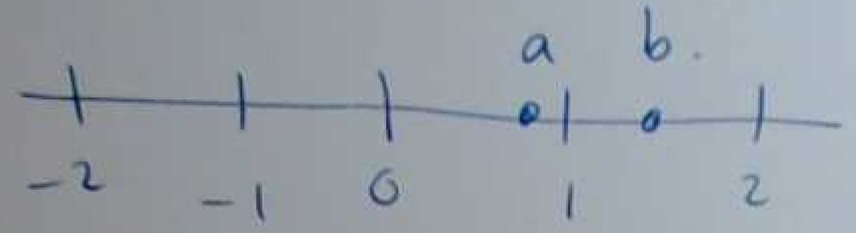
orbits of points. $\{0\}$.

$x \mapsto x + a$

$0 \mapsto a$.

orbit of $0 = \mathbb{R}$.

$\mathbb{R}$



$\mathrm{Isom}^+(\mathbb{R})$.

$x \longmapsto x+a$.

ordered pairs of points.

$$(a,b) \underset{\uparrow}{\overset{x \mapsto x-a}{\sim}} (0, b-a).$$

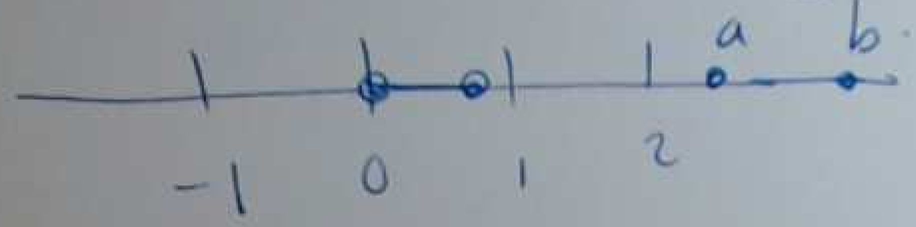
orbit equivalent

$\uparrow$ parameterization of pairs of points $b-a \in \mathbb{R}$.

$$\begin{array}{ccc} (a,b) & \sim & (c,d) \\ \downarrow & & \downarrow \\ (0,b-a) & & (0,d-c) \end{array} \leftarrow \text{same iff } b-a = d-c.$$

$$(0,1) \not\sim (1,0)$$

$\mathbb{R}$.



$\mathrm{Isom}^+(\mathbb{R}) \subset \mathrm{Sim}^+(\mathbb{R})$.

$x \longmapsto ax + b$.

~~$a \neq 0$~~ $a > 0$. $a, b \in \mathbb{R}$.

orbit$(0) = \mathbb{R}$.

ordered pairs of distinct points. $(a,b) \sim (0, b-a)$

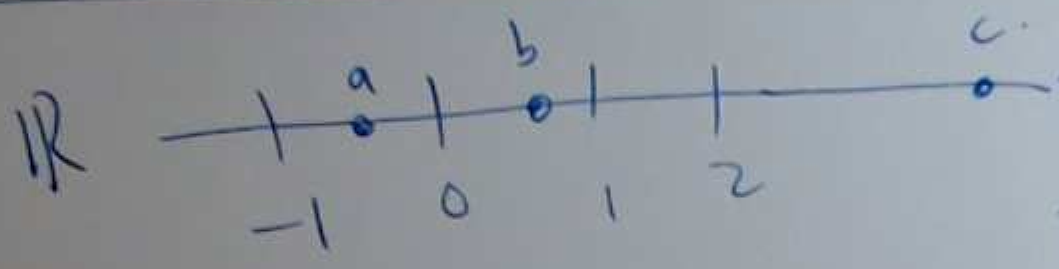
$x \mapsto x - a$.

$(a,b) \mapsto (0, b-a) \longmapsto \left(0, \frac{b-a}{|b-a|}\right) = (0,1)$ or $(0,-1)$

$x \mapsto x - a$ $\quad$ $x \mapsto \frac{x}{|b-a|}$

two case $\quad a < b \Leftrightarrow \sim (0,1)$

$a > b \sim (0,-1) \sim (1,0)$.



$\mathrm{Sim}^+(\mathbb{R})$

$x \mapsto ax+b \quad a>0$

unordered distinct triples $(0,1)$

$\{a,b,c\}$ write them as (a,b,c) $\quad a<b<c$

$(a,b,c) \rightsquigarrow (0, \underset{>0}{b-a}, c-a)$

$x \mapsto x-a$

parametrization of triples ✓

$\rightsquigarrow$

$x \mapsto \frac{x}{b-a}$ $\quad \left(0, 1, \underset{\in \mathbb{R}}{\frac{c-a}{b-a}}\right)$

$(0,1,s) \overset{?}{\sim} (0,1,t)$

$x \mapsto ax$ fixes 0, fix $1 \Rightarrow a=1$, $x \mapsto x$

$x \mapsto a(x-1)+1$ fix 1 $\quad 0 \mapsto 1-a=0$, $a=1$, $x \mapsto x$

Laurent series

$$\sum_{n \in \mathbb{Z}} c_n (z - z_0)^n$$

Aside

$$f(z) = e^{\sin(z)} \leftarrow \text{this has a power series.}$$

$$f'(z) = e^{\sin(z)} \cdot \cos(z)$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

last time

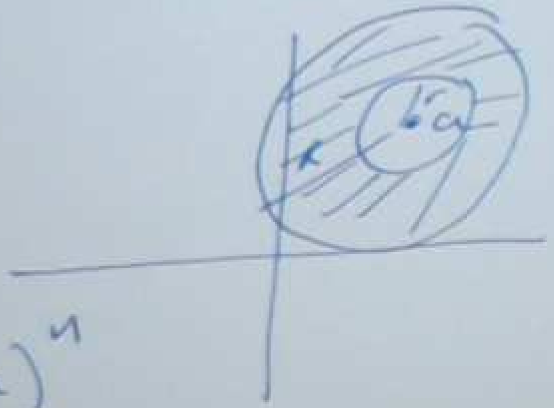
$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

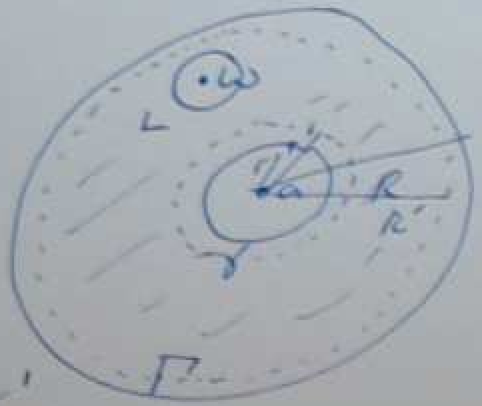
Thm Let K be an annulus $r < |z-a| < R$.

Let $f(z)$ which is analytic in K.

Then $f(z)$ has a Laurent series which is equal to $f(z)$.

$$\sum_{n \in \mathbb{Z}} c_n (z-a)^n.$$





Proof pick $w \in K$.

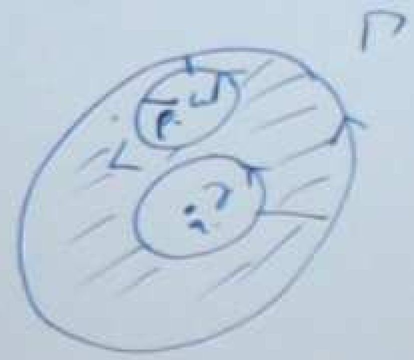
pick $r < r'$ $R' < R$

$w \in K' = r' < |z-a| < R'$

L small circle around w, contained in K'

$\Gamma = |z-a| = R'$ $\gamma = |z-a| = r'$ $\frac{f(z)}{z-w}$ analytic in Γ, outside γ, L.

$\dfrac{f(z)}{z-w}$ analytic inside Γ outside γ, L.



so $$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz + \underbrace{\frac{1}{2\pi i}\int_L \frac{f(z)}{z-w}\,dz}_{\text{Cauchy's integral formula}}$$

$= f(w)$.

$$f(w) = \underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz$$ ← write these as power series.

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)} = \frac{1}{(z-a)}\left(1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+\cdots\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n$$ here $\left|\frac{w-a}{z-a}\right| < \frac{\rho}{R'} < 1$.

<u>convergent</u>

$$f(w) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{\boxed{z-w}}\,dz + \boxed{\frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{\boxed{w-z}}\,dz} \qquad \frac{1}{z-w} \quad \frac{1}{w-z}$$

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)} = \frac{1}{w-a}\left(1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\dots\right)$$

$$= \sum_{n=1}^{\infty} \frac{(w-a)^{-n}}{(z-a)^{-n+1}}$$ converges if $\frac{|z-a|}{|w-a|} < 1$ ✓ $\frac{r'}{r'+\rho} < 1$

$$f(w) = \frac{1}{2\pi i}\int_{\Gamma} f(z)\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}\,dz + \frac{1}{2\pi i}\int_{\gamma} f(z)\sum_{n=1}^{\infty} \frac{(w-a)^{-n}}{(z-a)^{-n+1}}\,dz$$

integrate term by term

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n + \sum_{n=1}^{\infty} c_n (w-a)^{-n} \qquad \square$$

Suppose $|f(z)| \le M$ for all z in $r < |z-a| < R$.

then Cauchy inequality gives $|c_n| \le \dfrac{M}{2\pi} \dfrac{2\pi\rho}{\rho^{n+1}} = \dfrac{M}{\rho^{n+1}}$

$r < \rho < R$

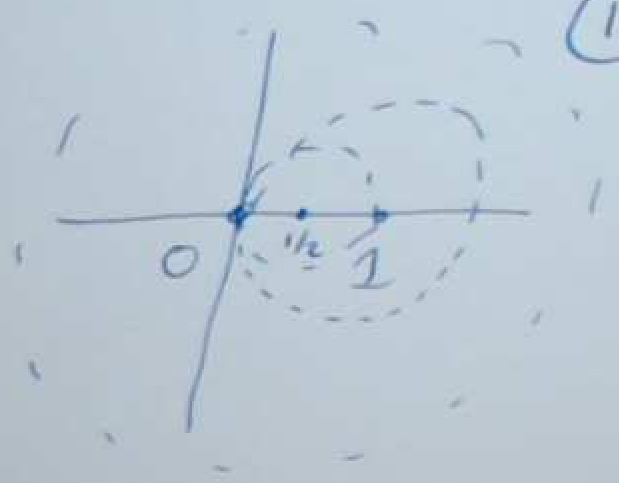
<u>Example</u> $f(z) = \dfrac{1}{z(1-z)}$

$0 < |z| < 1$ $\quad = \dfrac{1}{z} + \dfrac{1}{1-z}$

$f(z) = \dfrac{1}{z} + 1 + z + z^2 + z^3 + \cdots$ } Laurent series at $z=0$

converges in $0 < |z| < 1$

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$



in $0 < |z-1| < 1$.

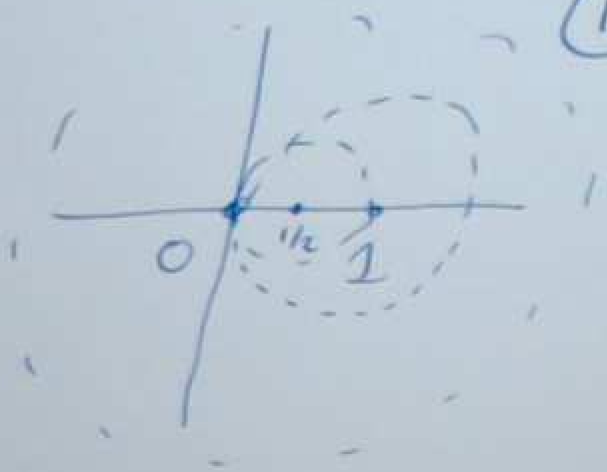
$$f(z) = \frac{1}{z-1} + \frac{1}{1+(z-1)}$$

$$= \frac{1}{z-1} + 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

Isolated singular points

Let $f(z)$ be analytic in a neighbourhood of z_0 but not defined at z_0. Then z_0 is an isolated singularity for $f(z)$.

$$f(z) = \frac{1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

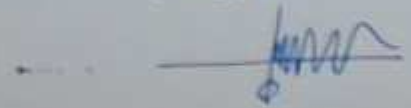


in $0 < |z-1| < 1$.

$$f(z) = \frac{1}{z-1} + \frac{1}{1+(z-1)}$$

$$= \frac{1}{z-1} + 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

$$\frac{1}{\sin\left(\frac{1}{z}\right)}$$



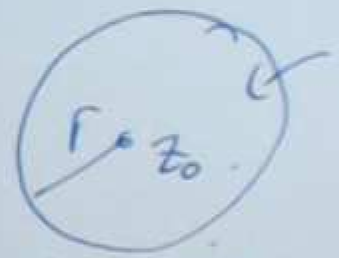
Isolated singular points

Let $f(z)$ be analytic in a neighbourhood of z_0 but not defined at z_0. Then z_0 is an <u>isolated singularity</u> for $f(z)$.

we have shown $f(z)$ has a Laurent series.

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n \quad (*)$$

in some deleted neighbourhood of z_0.



analytic in $0 < |z_0 - z| < r$

There are 3 possibilities.

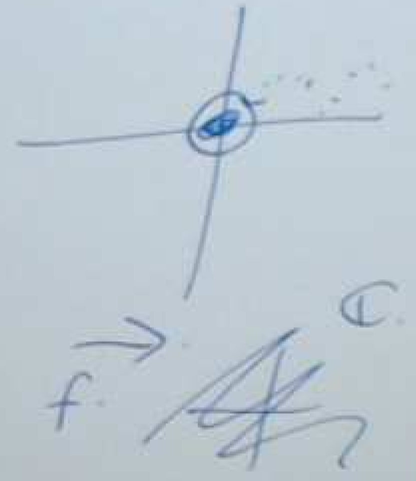
1) $(*)$ contains no negative powers
removable singularity

2) $(*)$ contains only finitely many negative powers.
pole

3) $(*)$ contains infinitely many negative powers
essential singularity

Motivation spse $f(z)$ has a pole at $z=0$ of order m.

$$\frac{c_m}{z^m} + \frac{c_{m-1}}{z^{m-1}} + \cdots c_0 + c_1 z + \cdots \qquad c_m \neq 0$$



then $\frac{1}{f(z)}$ has a zero of order m at 0.

then $f(z) \to \infty$ as $z \to 0$.

essential singularity $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \frac{1}{3! z^3} + \cdots$ ← infinitely many terms

$\lim_{z \to 0} e^{1/z}$ DNE. for any complex number $a \in \mathbb{C} \cup \{\infty\}$

there is a sequence of points $z_n \to 0$

s.t. $f(z_n) \to a$.

Removable singularities

$f(z)$ = power series. analytic in the full neighbourhood $|z| < r$.

$f(z) = \cancel{c_0 + c_1 z + c_2 z^2 + \cdots}$ set $f(z_0) = c_0$ ✓

$c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots$

Poles z_0 is a pole, Laurent series contains only finitely many negative powers.

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0) + \frac{c_{-1}}{z - z_0} + \cdots + \frac{c_{-m}}{(z - z_0)^m}$$

$\neq 0$

Removable singularities

$f(z) =$ power series analytic in the full neighbourhood $|z| < r$.

$$f(z) = \cancel{c_0 + c_1 z + c_2 z^2 + \cdots} \quad \text{set } f(z_0) = c_0 \checkmark$$

$$c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

Poles z_0 is a pole, Laurent series contains only finitely many negative powers.

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0) + \frac{c_{-1}}{z-z_0} + \cdots + \frac{c_{-m}}{(z-z_0)^m} \quad \neq 0$$

call this a pole of order m.

$m = 1$ simple pole

$m \geq 2$ multiple pole

<u>note</u>

$$(z-z_0)^m f(z) = \underbrace{\sum_{n=0}^{\infty} c_n (z-z_0)^{n+m} + c_{-1}(z-z_0)^{m-1} + \dots + c_{-m}}_{\text{power series}} \quad c_{-m} \neq 0$$

so z_0 is a removable singularity for $(z-z_0)^m f(z)$.

$$\lim_{z \to z_0} (z-z_0)^m f(z) = c_{-m} \neq 0.$$

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{c_{-m}}{(z-z_0)^m} = \infty.$$

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Thm Let z_0 be a zero of order m for a function $f(z)$ analytic at z_0. Then $\frac{1}{f(z)}$ is analytic in a deleted neighbourhood of z_0 and has a pole of order m at z_0.

Proof $f(z)$ has a power series $c_m(z-z_0)^m + c_{m+1}(z-z_0)^{m+1} + \cdots$

equivalently $f(z) = (z-z_0)^m g(z)$ $g(z_0) = c_m \neq 0$.

↑ analytic function in nbhd of z_0.

$$\frac{1}{f(z)} = \frac{1}{(z-z_0)^m} \cdot \frac{1}{g(z)} = \frac{h(z)}{(z-z_0)^m}$$

$h(z)$ is analytic and non-zero at z_0.

$\frac{1}{g}$

Joseph Maher

Joseph Maher

$f(z)$ is analytic in a deleted nbd of z_0

$(z-z_0)^m f(z)$ is analytic in full nbd $\Rightarrow$ z_0 is a pole of order m. $\square$

Thm Let z_0 be a pole of order m for a function $f(z)$ analytic in a deleted nbd of z_0.

Then $\frac{1}{f(z)}$ is analytic at z_0, provided $\frac{1}{f(z)} = 0$ with a zero of order m at z_0.

Proof $\square$.

Corollary $f(z)$ ~~non-vanishing~~ analytic? in a deleted nbd of z_0 and has an ess. sing at z_0, then $\frac{1}{f(z)}$ also has an essential singularity at z_0

$f(z)$ is analytic in a deleted nbd of z_0.

$(z-z_0)^m f(z)$ is analytic in full nbd $\Rightarrow$ z_0 is a pole of order m. $\square$

Thm Let z_0 be a pole of order m for a function $f(z)$ analytic in a deleted nbd of z_0.

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