(1) (10 points) Draw a picture of the complex number $1-i$ in the complex plane.

(a) Draw roughly where $(1-i)^3$ should be, and then find it exactly.
(b) Draw roughly where the cube roots of $1-i$ should be, and then find them exactly.
(c) Find e^{1-i}.
(d) Find all values of $\log(1-i)$.

(2) (10 points) Describe the geometric action of the following operations on the complex plane.

(a) $z \mapsto z+i$
(b) $z \mapsto iz$
(c) $z \mapsto i\overline{z}$
(d) $z \mapsto iz+i$

For the last two, describe them as isometries, not compositions of isometries.

(3) (10 points) Given a complex number a, let r_a be rotation by π about a.

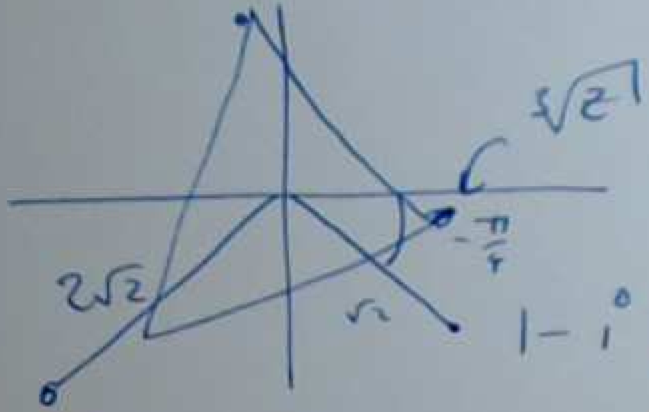
(a) Write down an explicit map giving r_a as a function $r_a \colon \mathbb{C} \to \mathbb{C}$.
(b) Show that $r_a r_b$ is a translation, and describe it explicitly in terms of a and b.

(4) (10 points) Find an analytic function that:

(a) takes the unit disc to the upper half plane
(b) takes the (open) first quadrant to the (open) lower half plane

Q1. $z = 1 - i$

a).



$\underbrace{e^{i\theta}}_{\text{polar}} = \underbrace{\cos\theta + i\sin\theta}_{\text{cartesian}}$

$1 - i \longleftrightarrow \sqrt{2}\, e^{-\frac{\pi}{4} i}$

$z^3 = 2\sqrt{2}\, e^{-\frac{3\pi}{4} i}$

$(1-i)^3 = 1 - 3i \cdots$

b) $z = \sqrt{2}\; e^{-\frac{\pi}{4} i}$

$2^{1/6}\, e^{-\frac{\pi}{12} i + \frac{2\pi n}{3}}$ $\quad n \in \mathbb{N}$

$2^{1/6}\, e^{-\frac{\pi}{12} i}\, e^{i\frac{2\pi n}{3}}$

$-\frac{\pi}{12}$

$\frac{7\pi}{12}$

$\frac{15\pi}{12}$

c) $e^{1-i} = e^1 e^{-i} = e\left(\cos(-1) + i\sin(-1)\right)$

$e\cos(1) - e\sin(1)$

d) $\log(1-i)$. $\log(z)$. $\log(re^{i\theta})$

$\log(\sqrt{2}) + i \cdot -\frac{\pi}{4}$ $= \log(r) + i\theta + 2\pi n$.

$\frac{1}{2}\ln(2) - \frac{\pi}{4}i + 2\pi n i$ $n \in \mathbb{Z}$. $i = e^{\frac{\pi}{2}i}$

Q2 a) $z \mapsto z+i \leftarrow$ translation by i $\mathbb{C}$.

b) $z \mapsto iz \leftarrow$ rotation by $\frac{\pi}{2}$.

c) $z \mapsto i\bar{z} \leftarrow$ reflection in $y=x$ $e^{\pi/2\, i}\bar{z}$ $\cancel{\pi/4}$.

d) $z \mapsto iz+i$.

$f = \bar{z}$

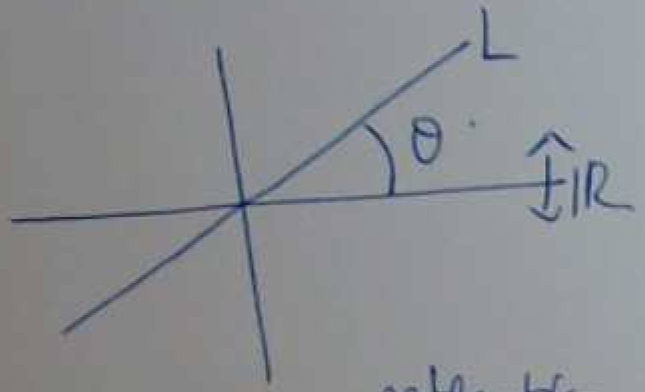
$g = e^{i\theta}z$.

gfg^{-1}

$z \mapsto e^{i\theta}\overline{e^{-i\theta}z} = e^{2i\theta}\bar{z}$

$z \mapsto i\bar{z}$ ← reflection.

$f(z) = \bar{z}$



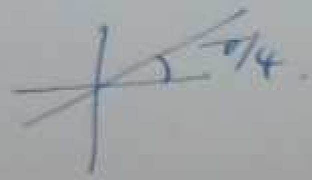
$g(z) = e^{i\theta} z$.

$g^{-1}(z) = e^{-i\theta} z$

reflection in L. → $g f g^{-1}(z)$

$$z \xmapsto[g^{-1}]{} e^{-i\theta} z \xmapsto[f]{} \overline{e^{-i\theta} z} = e^{i\theta} \bar{z} \xmapsto[g]{} e^{i\theta} e^{i\theta} \bar{z}$$

$r_L(z) = e^{2i\theta} \bar{z}$. $i\bar{z}$ $e^{\frac{\pi}{2} i} \bar{z}$



$z \mapsto -\bar{z}$ $-1 = e^{\pi i}$ reflection in $\theta = \frac{\pi}{4}$

Q3

r_a = rotation by π about a.

$f(z)$  $= -z$

$e^{i\theta} z$

$e^{i\pi} z = -z$

a) $g f g^{-1}$

$g(z) = z + a$

$g^{-1}(z) = z - a$

$$z \xmapsto{g^{-1}} z - a \xmapsto{f} -z + a \xmapsto{g} -z + 2a$$

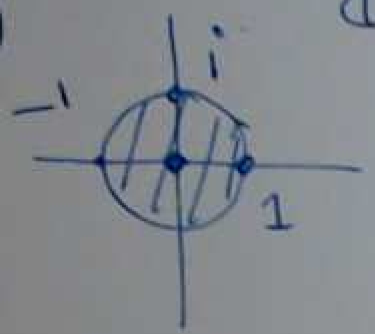
$r_a(z) = -z + 2a$

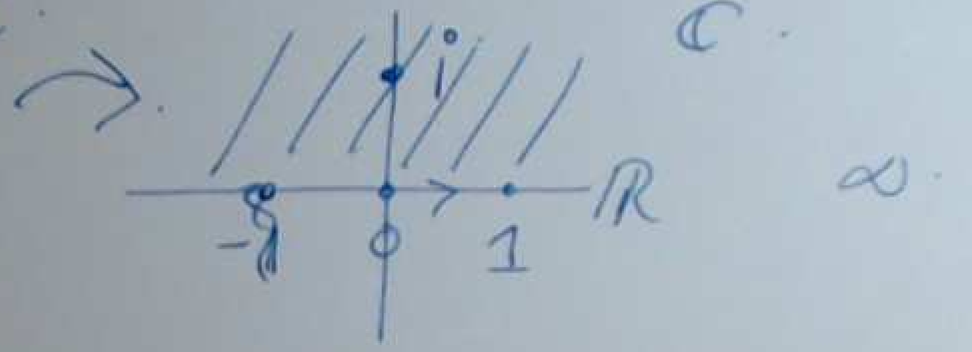
b) $r_a r_b$

$$z \xmapsto{r_b} -z + 2b \xmapsto{r_a} -(-z+2b) + 2a$$

$$z + 2(a-b)$$

Q4. a)





$1 \longmapsto 0$

$i \longmapsto 1$

$-1 \longmapsto \infty$

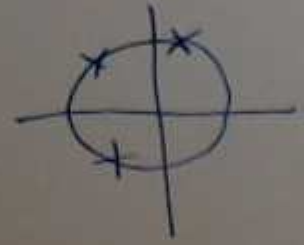
$$\boxed{f(z) = \frac{z-1}{z+1}\,\frac{i+1}{i-1}}$$

$0 \mapsto -\frac{i+1}{i-1}\,(-i-1)$

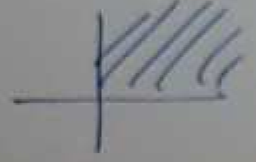
$\frac{1+i}{1-i}\,\frac{1+i}{1+i}$

$\frac{2i}{2} = i$

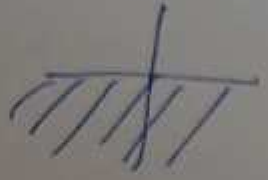
$$f(i) = \frac{i-1}{i+1} \times \frac{i+1}{i-1}$$

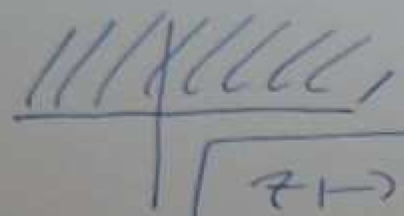


b)



$z \mapsto z^2$





$$\boxed{z \mapsto -i z^2}$$

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<u>Q5</u> $f(z) = \dfrac{\bar{z}}{|z|^2} = \dfrac{x - iy}{x^2+y^2} = \underbrace{\dfrac{x}{x^2+y^2}}_{u} + i\underbrace{\dfrac{-y}{x^2+y^2}}_{v}$

$\dfrac{\partial u}{\partial x} = u_x = \dfrac{(x^2+y^2) - x\,2x}{(x^2+y^2)^2} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$ ✓

$v_y = \dfrac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

$u_y = -\dfrac{x}{(x^2+y^2)^2}(2y) = \dfrac{-2xy}{(x^2+y^2)^2}$ $\quad v_x = -\dfrac{-y}{(x^2+y^2)^2}\cdot(2x)$

$f(z) = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{1}{z}$ $\quad = +\dfrac{2xy}{(x^2+y^2)^2}$ ✓

Q6 $u(x,y) = e^{x}\sin y$.

$u_x = e^{x}\sin y \qquad u_y = e^{x}\cos y$

$u_{xx} = e^{x}\sin y \qquad u_{yy} = -e^{x}\sin y \qquad u_{xx}+u_{yy}=0$ ✓

$u_x = v_y$

$v_x = -u_y$

$v = \int u_x\,dy = -e^{x}\cos y + f(x)$

$= -\int u_y\,dx = -e^{x}\cos y + f(y)$

$v = -e^{x}\cos y + c$

$f(x,y) = e^{x}\sin y + i(-e^{x}\cos y). = e^{x}(\sin y - i\cos y)$

$-i\cdot i$

$\boxed{-ie^{z}} = e^{x}(i\cos y + i\sin y)$

$$f(x,y) = u + iv = e^x \sin y - ie^x \cos y = -ie^z.$$

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y).$$

Q7 $\int_c \frac{1}{z^2}\,dz.$

$c(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$.

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \frac{1}{(c(\theta))^2}\, c'(\theta)\, d\theta = \int_0^{2\pi} \frac{ie^{i\theta}}{(e^{i\theta})^2}\, d\theta.$$

$$= i\int_0^{2\pi} \frac{1}{e^{i\theta}}\, d\theta = i\int_0^{2\pi} \frac{1}{e^{i\theta}} \frac{e^{-i\theta}}{e^{-i\theta}}\, d\theta = i\int_0^{2\pi} e^{-i\theta}\, d\theta$$

$$= i\int_0^{2\pi} \cos\theta - i\sin\theta\, d\theta = 0.$$

Q9 a) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} z^2$ b) $\sum_{n=1}^{\infty} \frac{z^n}{n^{n/2}}$ c) $\sum_{n=0}^{\infty} z^{n^2}$.

$\limsup \sqrt[n]{|c_n|} = \ell$ radius of convergence is $R = \frac{1}{\ell}$.

a) $\sqrt[n]{\frac{n^2}{2^n}} = \frac{1}{2}\sqrt[n]{n^2} = \frac{1}{2} n^{2/n} = \frac{1}{2} \underbrace{e^{\frac{2\log(n)}{n}}}_{e^0 = 1}$, $\frac{2\log(n)}{n} \to 0$.

$= \frac{1}{2}$

$\ell = \frac{1}{2}$ $R = 2$.

b) Warning $\lim_{n\to\infty} n^{1/n} = 1$.

$\lim_{n\to\infty} f(n)^{1/n} = ?$

b) $\cancel{\sqrt[n]{n^{n/2}} = n^{1/2} \quad \lim_{n\to\infty} \sqrt{n} = \infty.}$

$\ell = \sqrt[n]{\frac{1}{n^{n/2}}} = \frac{1}{n^{1/2}} = \frac{1}{\sqrt{n}} \longrightarrow 0$ as $n \to \infty$.

$\ell = 0 \Rightarrow R = \infty$.

c) $\sum_{n=0}^{\infty} z^{n^2}$. $\quad 1 + z + z^4 + z^9 + \cdots$

$a_n = 0, 1$.

$\limsup \underbrace{\sqrt[n]{|a_n|}}_{0,1} = 1$.

$\ell = 1 \quad R = 1$.

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b) $\cancel{\sqrt[n]{n^{n/2}} = n^{1/2} \quad \lim_{n\to\infty} \sqrt{n} = \infty.}$

$\sqrt[n]{\dfrac{1}{n^{n/2}}} = \dfrac{1}{n^{1/2}} = \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.

$l = 0 \Rightarrow R = \infty$

c). $\displaystyle\sum_{n=0}^{\infty} z^{n^2}$ $\qquad 1 + z + z^4 + z^9 + \cdots$

$a_n = 0, 1$.

$\limsup \underbrace{\sqrt[n]{|a_n|}}_{0,1} = 1$.

$l = 1 \quad R = 1$.

$n^{1/n}$

Q10 $\cos 3\theta$. $\qquad e^{i\theta} = \cos\theta + i\sin\theta$.

$$e^{i3\theta} = \cos 3\theta + i\sin 3\theta.$$

$\|$

$$(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta \cdot -\sin^2\theta - i\sin^3\theta.$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

$=$

$1 - \cos^2\theta$.

$$= 4\cos^3\theta - 3\cos\theta \checkmark$$

recall

§11 Laurent series.

$$\sum_{n\in\mathbb{Z}} c_n (z-a)^n = \underbrace{\sum_{n=0}^{\infty} c_n(z-a)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}}_{\text{principal part}}$$

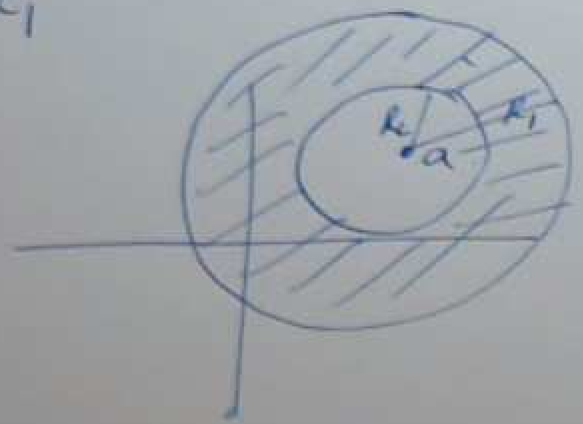
radius of convergence:

$$\frac{1}{\limsup \sqrt[n]{|c_n|}} \qquad\qquad \limsup \sqrt[n]{|c_n|}$$

$$R_1 \qquad\qquad R_2$$

↑

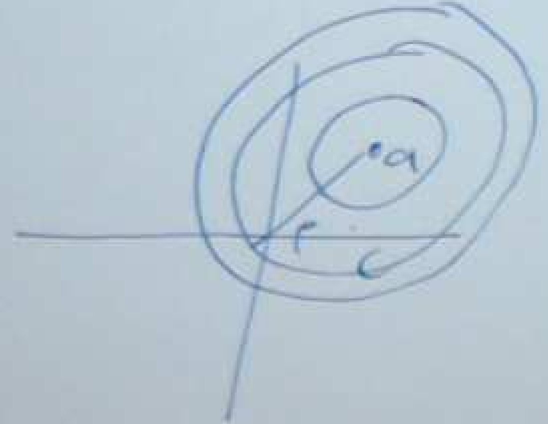
$\min\{R_1, R_2\}$



always assume

$R_1 > R_2$

useful facts: Laurent series is uniformly convergent in every closed bounded subdomain $R_2 < |z-a| < R_1$.



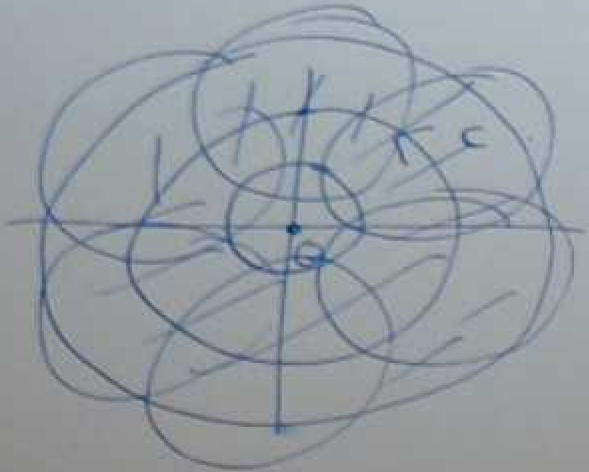
<u>Thm</u> Let C be any circle in $|z-a| = \rho$. $R_2 < \rho < R_1$

then

$$c_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}} \, dz \qquad n \in \mathbb{Z}$$

<u>motivation</u> $f(z) = z + \frac{1}{z}$

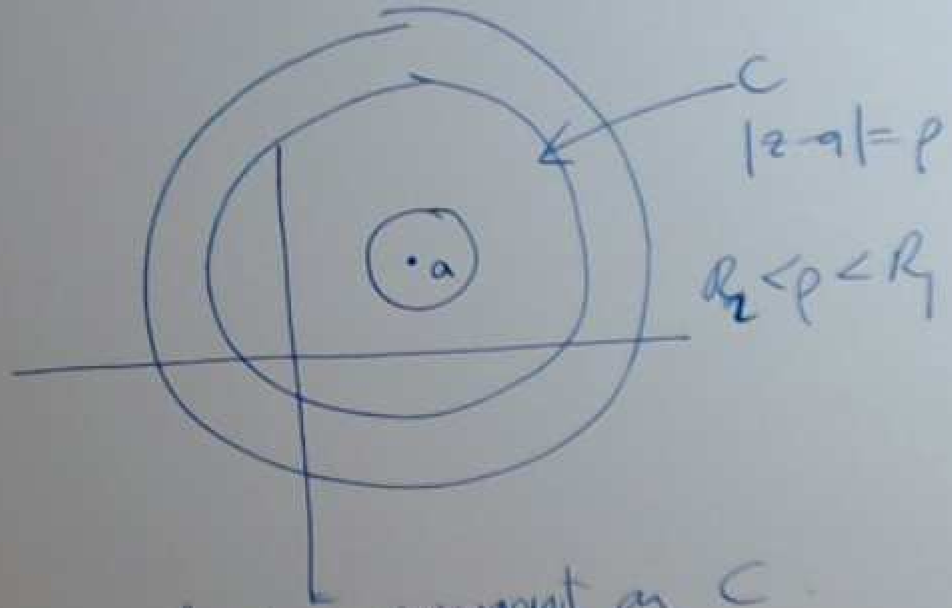
$$\sum_{n=-\infty}^{\infty} c_n z^n$$



$\mathbb{C}$

Thm

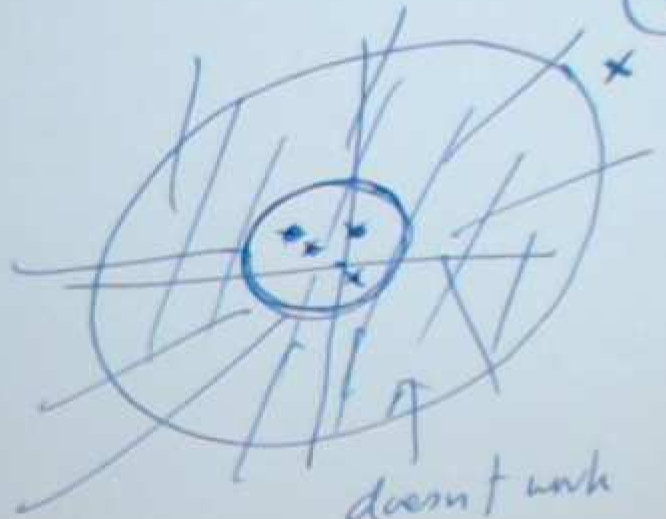
$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$



series is uniformly convergent on C.

$$\frac{1}{2\pi i}\frac{f(z)}{(z-a)^{n+1}} = \sum_{k=-\infty}^{\infty} c_k \frac{(z-a)^k}{(z-a)^{n+1}}$$

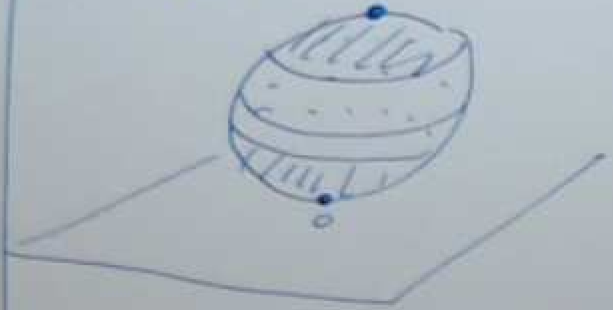
still unif conv on C



doesn't work

$$\sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \frac{c_n}{z^n}$$

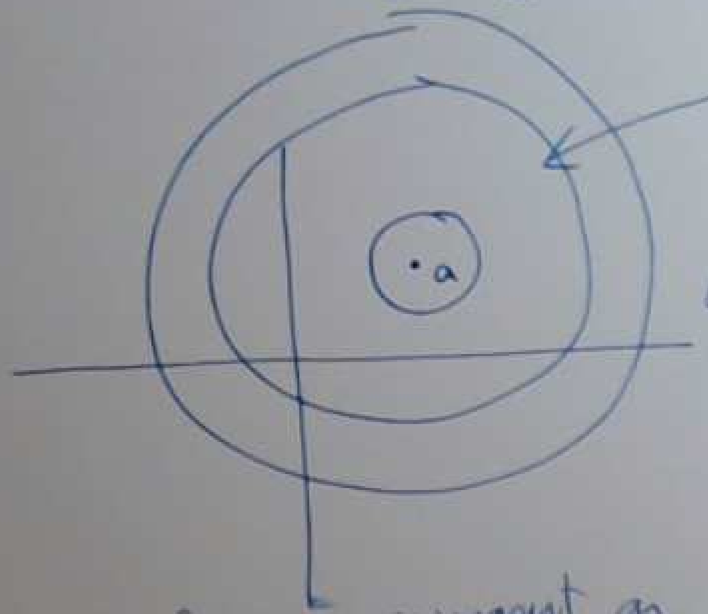
∞



$f(z)$

Thm

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$



series is uniformly convergent on C

$$\frac{1}{2\pi i}\frac{f(z)}{(z-a)^{n+1}} = \sum_{k=-\infty}^{\infty} c_k \frac{(z-a)^k}{(z-a)^{n+1}}$$

still unif conv. on C

can integrate this term by term.

doesn't work

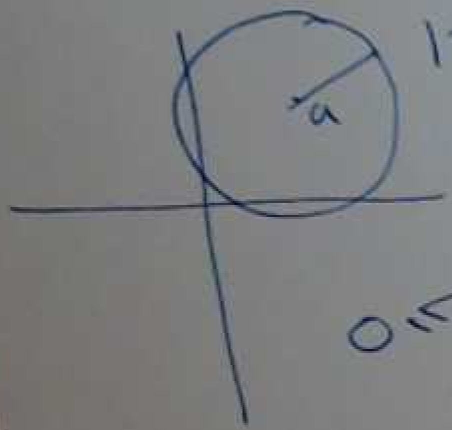
$$\sum_{n=0}^{\infty} c_n z + \sum_{n=1}^{\infty} \frac{c_n}{z^n}$$

$f(z)$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_C (z-a)^{k-n-1}\,dz.$$

$$\uparrow \infty$$

$$= \sum_{k=-\infty} c_k \frac{1}{2\pi i}\left(\rho e^{i\theta}\right)^{k-n-1} c'(\theta)\,d\theta.$$



$|z-a| = \rho.$

$z-a = \rho e^{i\theta}.$

$z = a+\rho e^{i\theta}.$

$0 \leq \theta \leq 2\pi$

$c(\theta) = a+\rho e^{i\theta}.$

$c'(\theta) = i\rho e^{i\theta}.$

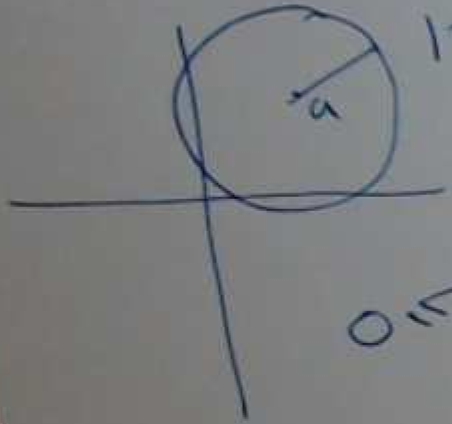
$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_0^{2\pi} \rho^{k-n-1} e^{i\theta(k-n-1)}\, i\rho e^{i\theta}\,d\theta$$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} \rho^{k-n} e^{i\theta(k-n)}\,d\theta = 0$$

except when $k=n$ $\square$.

$= c_k = c_n.$

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_C (z-a)^{k-n-1}\,dz.$$



$|z-a| = \rho.$

$z-a = \rho e^{i\theta}.$

$z = a+\rho e^{i\theta}.$

$0 \le \theta \le 2\pi$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\left(\rho e^{i\theta}\right)^{k-n-1} c'(\theta)\,d\theta.$$

$c(\theta) = a+\rho e^{i\theta}.$

$c'(\theta) = i\rho e^{i\theta}.$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_0^{2\pi} \rho^{k-n-1} e^{i\theta(k-n-1)}\, i\rho e^{i\theta}\,d\theta$$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} \rho^{k-n} e^{i\theta(k-n)}\,d\theta = 0$$

except when $k=n$

$$= c_k = c_n.$$

$\square$