$z = x + iy$

$\cos(5-i)$

e^z

$$\boxed{e^z = e^x(\cos y + i\sin y)}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\left.\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ \sin(z) &= \cancel{4}\ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \end{aligned}\right\}$$

Def^{n}

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$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

$$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} \cdots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \cos(z) + i\sin z$$

$$\boxed{e^{iz} = \cos(z) + i\sin z}$$

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$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\cdots$$

$$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} \cdots\cdots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\right)$$

$$= \cos(z) + i\sin z$$

$$\boxed{e^{iz} = \cos(z) + i\sin z}$$

$$e^{iz} = \cos(z) + i\sin(z)$$

$$e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z)$$

$$e^{iz} + e^{-iz} = 2\cos(z) \qquad \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\boxed{e^{x+iy} = e^{x}(\cos y + i\sin y)} \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos(5-i) = \frac{e^{i(5-i)} + e^{-i(5-i)}}{2}$$

$$= \frac{e^{1+5i} + e^{-1-5i}}{2} = \frac{1}{2}\Big[e(\cos 5 + i\sin 5) + e^{-1}(\cos(-5) + i\sin(-5))\Big]$$

$$= \frac{1}{2}\left(\left(e+\frac{1}{e}\right)\cos(5) + i\left(e-\frac{1}{e}\right)\sin(5)\right)$$

$$-1 \mapsto i$$
$$\infty \mapsto 1$$
$$i \mapsto 1+i$$

$$a \mapsto 0$$
$$b \mapsto \infty$$
$$c \mapsto 1$$

$$\boxed{\frac{z-a}{z-b} \cdot \frac{c-b}{c-a}}$$

$$-1 \mapsto 0$$
$$\infty \mapsto \infty \qquad f(z) = \frac{z+1}{1+i}$$
$$i \mapsto 1.$$

$$\boxed{g^{-1}(f(z)).}$$

$$i \mapsto 0$$
$$1 \mapsto \infty \qquad g(z) = \frac{z-i}{z-1} \cdot \frac{1+i-1}{1+i-i} = i\,\frac{z-i}{z-1}$$
$$1+i \mapsto 1$$

$$g(z) = \frac{iz+1}{iz-i}$$

$$-1 \mapsto i \qquad \infty \mapsto 1 \qquad i \mapsto 1+i$$

$$a \mapsto 0 \qquad b \mapsto \infty \qquad c \mapsto 1$$

$$\boxed{\frac{z-a}{z-b} \cdot \frac{c-b}{c-a}}$$

$$-1 \mapsto 0 \qquad \infty \mapsto \infty \qquad i \mapsto 1.$$

$$f(z) = \frac{z+1}{1+i}$$

$$\boxed{g^{-1}(f(z)).}$$

$$i \mapsto 0 \qquad 1 \mapsto \infty \qquad 1+i \mapsto 1$$

$$g(z) = \frac{z-i}{z-1} \cdot \frac{1+i-1}{1+i-i} = i\frac{z-i}{z-1}.$$

$$g(z) = \frac{iz+1}{iz-i}$$

$$f(z) = \frac{z+1}{1+i} \qquad \frac{z+1}{0z+(1+i)} \qquad g^{-1}(f(z)).$$

$$g(z) = \frac{iz+1}{iz-1} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$g^{-1}(z) = \frac{-z-1}{-iz+i}$$

$$\begin{bmatrix} -1 & -1 \\ -i & i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1+i \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

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$-1 \longmapsto 0$

$\infty \longmapsto \infty$

$i \longmapsto 1$

$a \longmapsto 0$

$b \longmapsto \infty$

$c \longmapsto 1$

$\frac{z-a}{z-b} \quad \frac{c-b}{c-a}$

$\frac{az+b}{cz+d}$

$f(z) = \frac{z+1}{0z+1} \quad \frac{1}{1+i}$

$-\frac{d}{c} \longmapsto \infty$

$f(z) = \frac{\cancel{z+1} \; \frac{1}{1+i} z + \frac{1}{1+i}}{0z+1}$

$\infty \longmapsto \infty$

$cz+d \neq 0$

$c=0 \quad d \neq 0$

$az+b$

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Q: $\sum_{n=0}^{\infty} c_n z^n$ when does it converge?

<u>recall</u> ratio test: $\cancel{\lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = \ell < \infty}$ ~~then it converges.~~

(everything $\in \mathbb{R}$)

$$\lim_{n \to \infty} \frac{|c_{n+1} z^n|}{|c_n z^n|} = \ell < 1$$ then it converges.

problem: limit might not exist

[Cauchy-Hadamard] $\limsup \sqrt[n]{|c_n|} = \ell.$

lim sup always exists! (may be ∞) $R = \frac{1}{\ell}.$

8

$$\sum_{n=0}^{\infty} z^{n^3} = 1 + z + z^8 + z^{27} + z^{64} + \dots$$

$$c_n = 1, 1, 0, 0, 0 \dots, \underset{c_8}{1}, 0, \dots 0, \underset{c_{27}}{1}, 0 \dots$$

$1 + z + z^4 + z^5 + \dots \leftarrow R = 1$

$\lim \left| \frac{c_{n+1}}{c_n} \right|$ DNE

$\limsup \sqrt[n]{|c_n|}$ $\quad 1, 1, 0, \dots 1, \dots 0, 1, 0 \dots$

$\limsup 1, 0, \dots 1, \dots = 1 = \ell$

$$R = \frac{1}{\ell}$$

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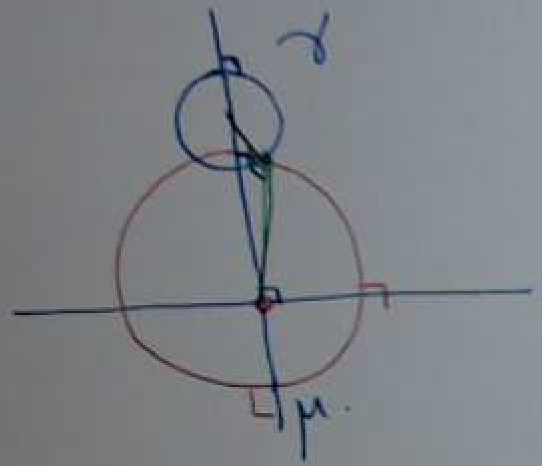
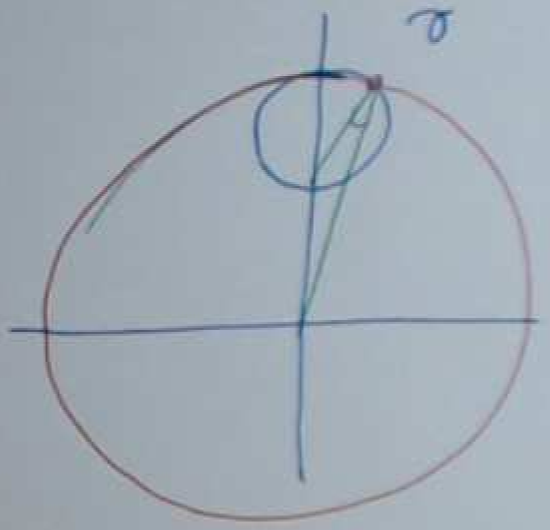
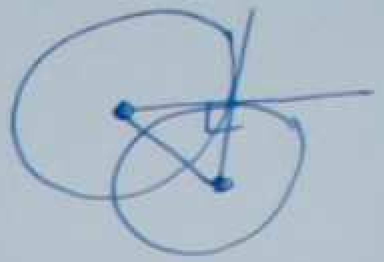
HW6. Q3. b)

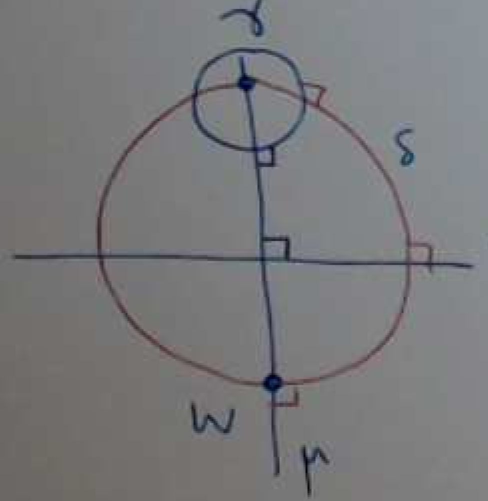
$\mathbb{C}$.



λ.





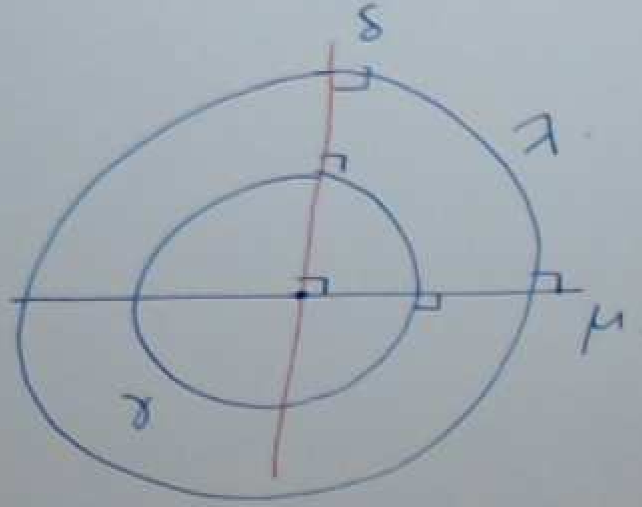


c)

f

Möbius map

$w \mapsto \infty$



λ, γ concentric

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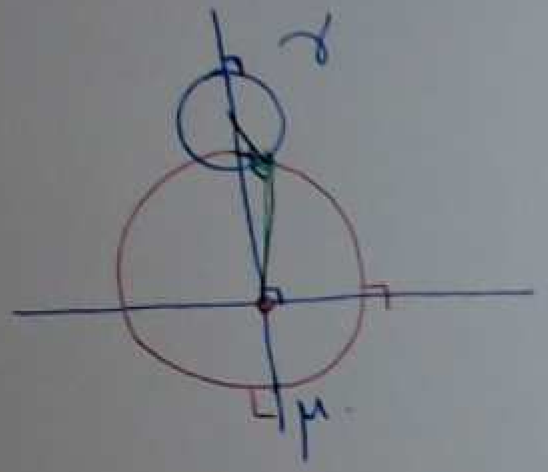
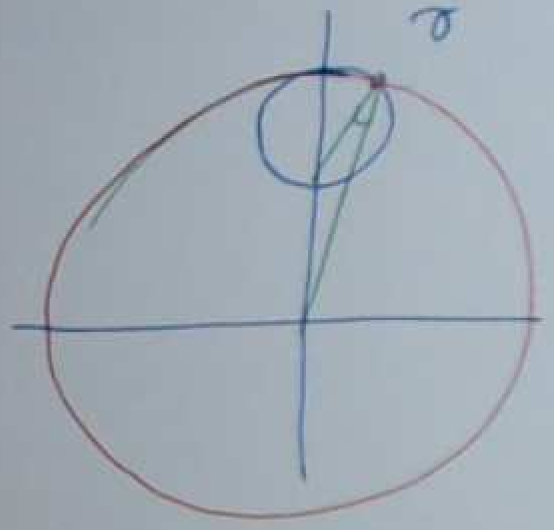
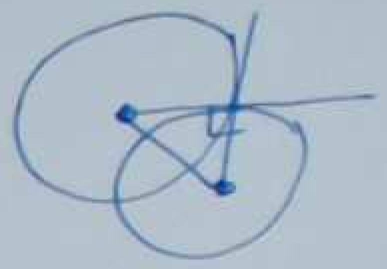
HW6. Q3 b)

$\mathbb{C}$

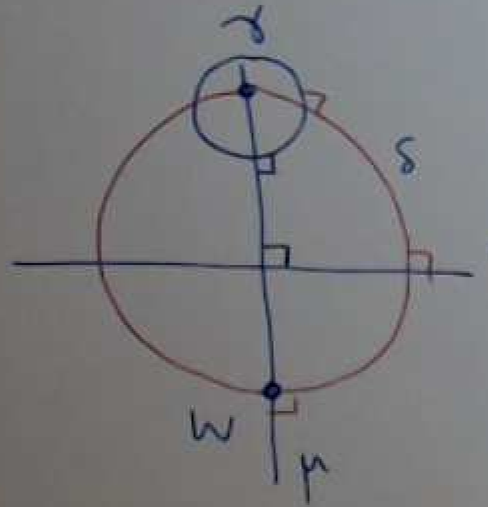


λ





∞



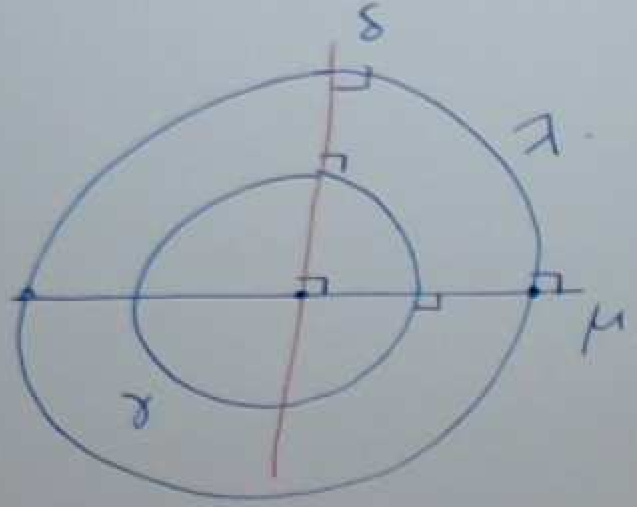
c)

f

Möbius map

$w \mapsto \infty$

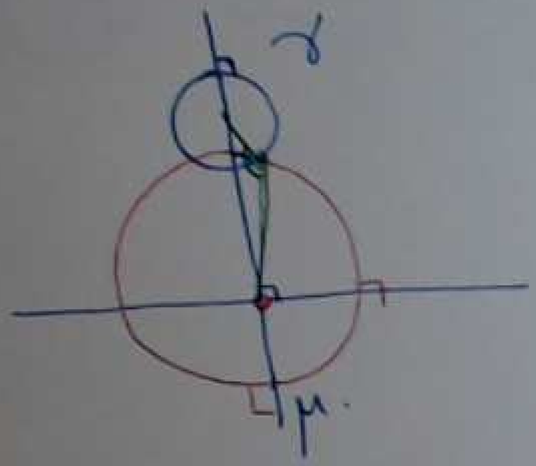
∞



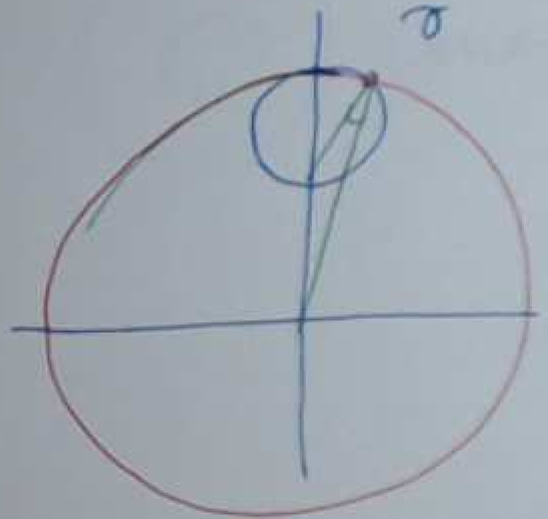
λ, γ concentric

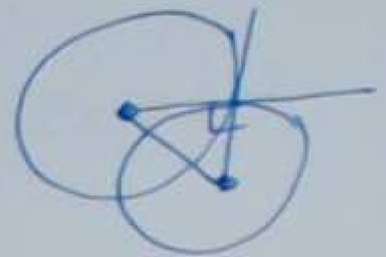
HW6. Q3. b)

$\mathbb{C}$

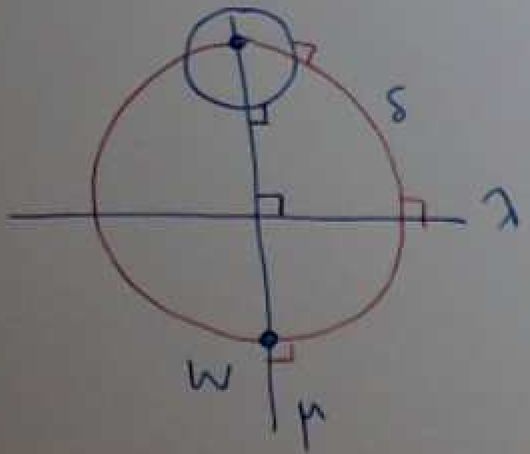


λ





∞



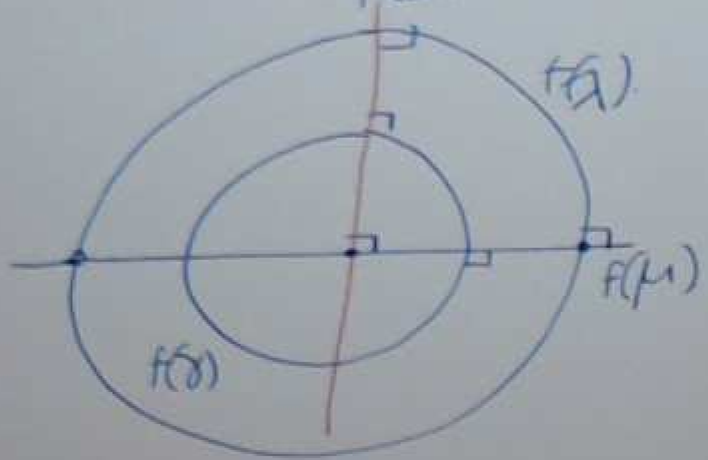
c)

f

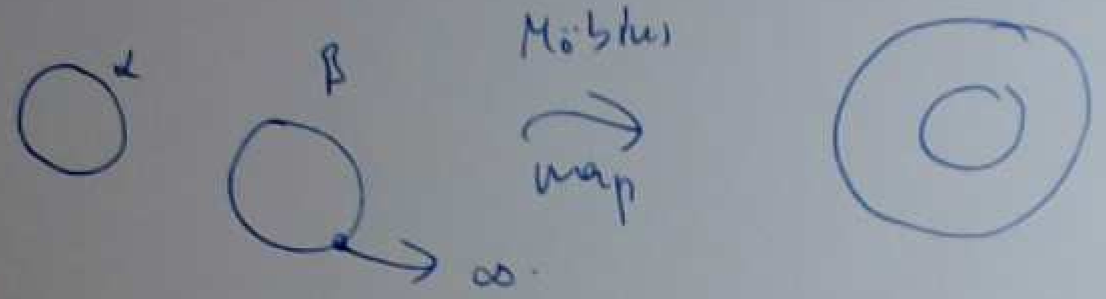
Möbius map

$w \mapsto \infty$

∞

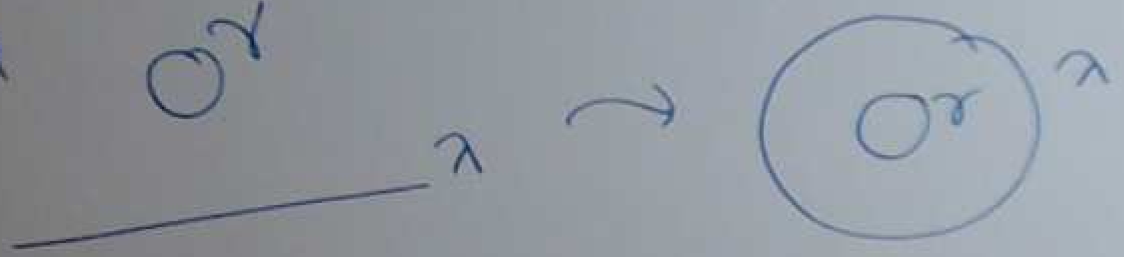


λ, γ concentric



concentric circles

showed



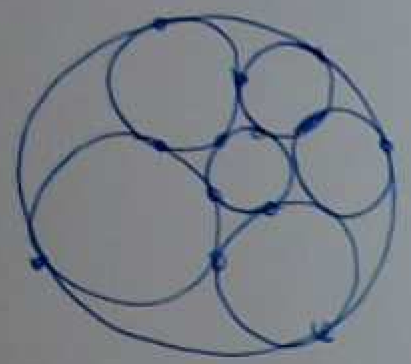
concentric.

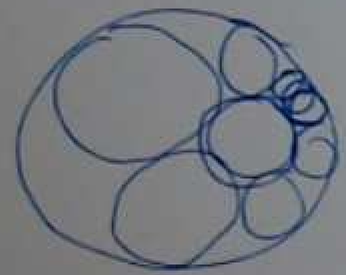
concentric

not disjoint.

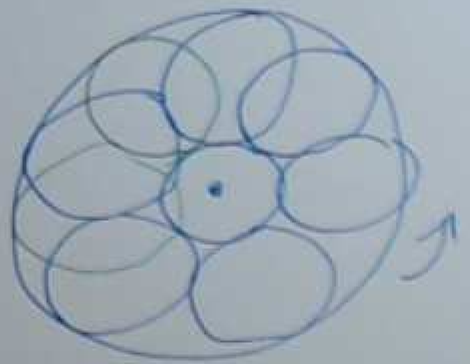
Steiner's Porism

if the circles match up then they always match regardless of the choice of initial circle.





Möbius map $\longrightarrow$



concentric circles.

tangent circles $\rightarrow$ tangent circles

$$\sum_{n=0}^{\infty} \frac{z^2}{n^2}$$

Q: radius of convergence ?

$$\limsup \sqrt[n]{|c_n|} = \limsup \sqrt[n]{\frac{1}{n^2}}$$

$$= \limsup e^{\log\left(\frac{1}{n^2}\right)\frac{1}{n}} = \limsup e^{-\underbrace{\frac{2\log(n)}{n}}_{\to 0 \text{ as } n\to\infty}}$$

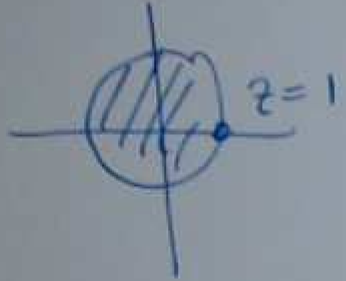
$$\ell = e^0 = 1 \qquad R = \frac{1}{\ell} = 1.$$

diverges $|z| > 1$

converges $|z| < 1$

Q: $|z| = 1$?

$$\sum_{n=0}^{\infty} \frac{z^n}{n^2}$$



$$\sum_{n=0}^{\infty} \frac{1}{n^2} \leftarrow \text{p-series convergent}$$

absolutely convergent

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \frac{\pi^2}{6}.$$

let $|z| \le 1$

$$\sum_{n=0}^{\infty} \frac{z^n}{n^2} \overset{\text{absolute}}{\underset{\text{series}}{\sim}} \sum_{n=0}^{\infty} \frac{|z|^n}{n^2} \le \sum_{n=0}^{\infty} \frac{1}{n^2}.$$

so abs. conv. for all $|z| \le 1$.

Q: uniform convergence $|z| \le 1$

for all $\epsilon > 0$ there is a N s.t. $|\underbrace{s(z)}_{\text{full sum}} - \underbrace{s_n(z)}_{\text{partial sum}}| < \epsilon$

for all $|z| \le 1$

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$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

$$\left| s(z) - s_n(z) \right|.$$

$$\left| \frac{z^{n+1}}{(n+1)^2} + \frac{z^{n+2}}{(n+2)^2} + \frac{z^{n+3}}{(n+3)^2} + \cdots \right|.$$

$|z| \leq 1.$

$$\leq \frac{|z|^{n+1}}{(n+1)^2} + \frac{|z|^{n+2}}{(n+2)^2} + \cdots \leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots$$

$\uparrow$

$|s(1) - s_n(1)|.$

$\sum \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges

$\Rightarrow |s(1) - s_n(1)| \longrightarrow 0$ $\leq \epsilon$

key point $\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots$ $\leftarrow$ does not depend on z, with $|z| \leq 1$.

$\longrightarrow 0$ as $n \to \infty$

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recall maximum moduli principles

- $f(z)$ analytic in a domain G $|f(z)|$ does not have a max in G

- $f(z)$ analytic in closed bounded domain G, cts on ∂G

  $|f(z)| = \text{const}$ on $\partial G \Rightarrow f(z) = \text{const}$

- $f(z)$ analytic in a bounded domain G then non-vanishing

  $|f(z)|$ attains its max and min on ∂G

Corollary Let f be analytic, non-vanishing on a bounded domain G. Then if $|f(z)|$ is constant on ∂G then f is constant in G.

Proof G bounded $\Rightarrow$ there are points $z_0, z_1 \in \partial G$ s.t. $|f(z_0)| = \min_{z \in \overline{G}} |f(z)|$ $\quad |f(z_1)| = \max_{z \in \overline{G}} |f(z)|$.

$|f(z)| = \text{const}$ on $\partial G \Rightarrow |f(z_0)| = |f(z_1)|$.

$\Rightarrow \max |f(z)| = \min |f(z)| \Rightarrow |f(z)| = \text{const}$

previous Thm $|f(z)| = \text{constant} \Rightarrow f(z) = \text{const}$ $\square$

Analogs for harmonic functions

<u>Thm</u> If $u(x,y)$ is harmonic, not constant in G domain
then $u(x,y)$ has ~~or~~ no max or min in G.

<u>Proof</u> Let f be an analytic with u as its real part
$f(z) = u + iv$, consider $g(z) = e^{f(z)}$ is analytic

$$|g(z)| = |e^{f(z)}| = |e^{u+iv}| = |e^{u(x,y)}|\,\underbrace{|\cos v + i\sin v|}_{1}$$

$$= |e^{u(x,y)}|.$$

$g(z)$ is non-vanishing
non-constant

$\Rightarrow$ $|g(z)|$ has no max or min in G.

$\Rightarrow$ $|e^{u(x,y)}|$ has no max/min in G $\Rightarrow$ $u(x,y)$ has no max or min in G $\square$

<u>Corollary</u> Let u be harmonic in a domain G

then ∂G has points z_0, z_1 s.t.

$$u(z_0) = \min_{(x,y) \in G} u(x,y) \qquad u(z_1) = \max_{(x,y) \in G} u(x,y) \qquad \square$$

<u>Corollary</u> Let u be harmonic on a bounded domain G, if u is constant on ~~G~~ ∂G, then $u = \text{const}$ in G. $\square$

§11 Laurent series ← power series in z and $\frac{1}{z}$.

Thm given a series $c_0 + \frac{c_1}{z-a} + \frac{c_2}{(z-a)^2} + \frac{c_3}{(z-a)^3} + \cdots$

let $l = \limsup \sqrt[n]{|c_n|}$

Then one of the following occurs:

1) if $l = 0$ the series is absolutely convergent for all z in $\mathbb{C} \cup \{\infty\} \setminus \{a\}$.

2) if $0 < l < \infty$ then the series is abs convergent for all $|z-a| > l$ and divergent for all $|z-a| < l$.

3) if $l = \infty$ the series diverges for all $z \in \mathbb{C} \setminus \{a\}$.

Proof

$$c_0 + \frac{c_1}{z-a} + \frac{c_2}{(z-a)^2} + \frac{c_3}{(z-a)^3} + \cdots$$

set $w = \frac{1}{z-a}$.

$$c_0 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots$$

← apply Cauchy–Hadamard to this series.

$\ell = \limsup \sqrt[n]{|c_n|}$

this has radius of converge $R = \frac{1}{\ell}$

converges	diverges
$\lvert w\rvert < \frac{1}{\ell}$	$\lvert w\rvert > \frac{1}{\ell}$
$\frac{1}{\lvert z-a\rvert} < \frac{1}{\ell}$	$\frac{1}{\lvert z-a\rvert} > \frac{1}{\ell}$
$\lvert z-a\rvert > \ell$	$\lvert z-a\rvert < \ell$

$\square$

Laurent series

$$(*) \quad \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{c_n}{(z-a)^n}}_{\text{principle part}}$$

suppose radius of convergence R $\qquad$ r

$$R = \frac{1}{\ell} = \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{|c_n|}}$$

(*) converges.

$$r < |z-a| < R \qquad r = \limsup_{n \to \infty} \sqrt[n]{|c_n|}.$$

assuming $r < R$

Furthermore: $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is

absolutely and uniformly convergent in any closed bounded subdomain $r < |z-a| < R$

and is analytic in

Example $z + \frac{1}{z}$ $0 < |z| < \infty$

Better example: $e^z + e^{1/z}$ $0 < |z| < \infty$

regular principle

$1 + z + \frac{z^2}{2!} + \cdots$ $1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots$

Furthermore: $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is

absolutely and uniformly convergent in any closed bounded subdomain $r < |z-a| < R$

and is analytic in

Example $z + \frac{1}{z}$. $0 < |z| < \infty$.

Better example: $e^z + e^{1/z}$ $0 < |z| < \infty$.

regular principle

$1 + z + \frac{z^2}{2!} + \cdots$ $1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots$

$f(z)$

Q: how to find c_n?

Thm Let C be the circle in $|z-a| = \rho$. $r < \rho < R$

Then $$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz. \quad n \in \mathbb{Z}.$$