§9.3 Riemann surfaces

$$\begin{tikzcd} G \arrow[r, "n-1"'] \arrow[dr, "1-1"'] & \mathbb{C} \arrow[l, bend right, dotted, "f^{-1}"'] \\ & S \arrow[u, "n-1"'] \end{tikzcd}$$

$f : G \to \mathbb{C}$. not univalent
injective.

$f : G \to f(G)$
univalent

$f'(z) \neq 0$

$f^{-1\prime}(z) = \dfrac{1}{f(w)} \neq 0.$

last time: $z \mapsto z^n$ $\quad z \mapsto e^z$.

- $\mathbb{C} \cup \{\infty\}$
- remark: different functions might need different cuts.

$f : \mathbb{C} \to \mathbb{C}$

$z \mapsto z^2 - 2z = w.$

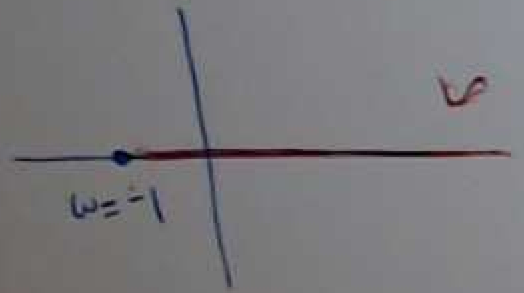
$\xleftarrow{f^{-1}}$

$f^{-1}(z) = 2 \pm \sqrt{4}$

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto z^2 - 2z$$

$$\begin{tikzcd} f^{-1}(w) & w \arrow[l, mapsto] \\ & S \arrow[ul, "f^{-1}"] \arrow[u, "2-1"'] \\ & \mathbb{C} \end{tikzcd}$$

$$z^2 - 2z = w$$
$$z^2 - 2z - w = 0$$
$$z = \frac{2 \pm \sqrt{4 + 4w}}{2} = 1 \pm \sqrt{1 + w}$$



w $[-1, \infty) \subseteq \mathbb{R}$

§ 10 Taylor series

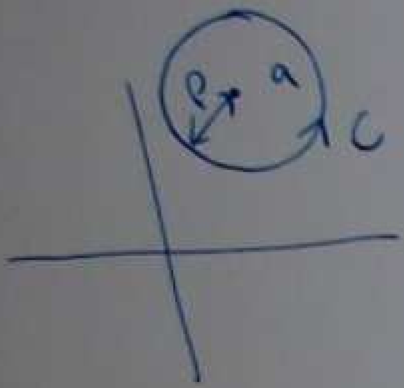
let $f(z)$ be analytic in a domain containing a, $f^{(n)}(a)$ exists for all n.

let $c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots$

where $c_n = \dfrac{f^{(n)}(a)}{n!}$

this is the Taylor series for f at a.

let R be the radius of convergence of the Taylor series, pick $\rho < R$.



$\rho < R$ radius of convergence.

$c_n = \dfrac{f^{(n)}(a)}{n!}$

Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

so $\quad c_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$

so if $|f(z)| \leq M$ for all $|z| < R$ then

$$|c_n| \leq \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+1}} = \frac{M}{\rho^n}$$

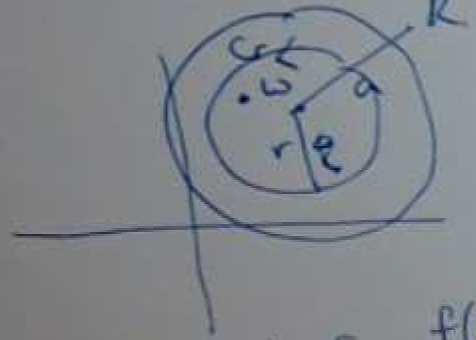
take limit as $\rho \to R$

$|c_n| \leq \dfrac{M}{R^n}$

Thm Let K be a disc $|z-a|<R$, $f(z)$ analytic in K. Then $f(z)$ has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof



$r<R$

$w \in K$. pick r big enough so C_r // C contains w.

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz \leftarrow \text{aim: write this as a power series}$$

$z \in C$ $$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}$$

$$= \frac{1}{z-a}\left(1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\left(\frac{w-a}{z-a}\right)^3+\cdots\right)$$

Thm Let K be a disc $|z-a|<R$, $f(z)$ analytic in K. Then $f(z)$ has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Proof

$r < R$.
$w \in K$. pick r big enough so C_r // C contains w.

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\, dz$$ ← aim: write this as a power series

$z \in C$ $$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}$$

$$= \frac{1}{z-a}\left(1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

$$\frac{1}{2\pi i}\frac{f(z)}{z-w} = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{f(z)(w-a)^n}{(z-a)^{n+1}}$$

set $M = \max_{z \in C} |f(z)|$

then $\left| \frac{1}{2\pi i} f(z) \frac{(w-a)^n}{(z-a)^{n+1}} \right| \leq \frac{M}{2\pi r} \underbrace{\left|\frac{w-a}{r}\right|}_{<1}^{n}$

so this sum is uniformly convergent on C ⇝ integrate term by term.

So $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dw = \sum_{n=0}^{\infty} c_n (w-a)^n$

so $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{f^{(n)}(a)}{n!}$ $\square$

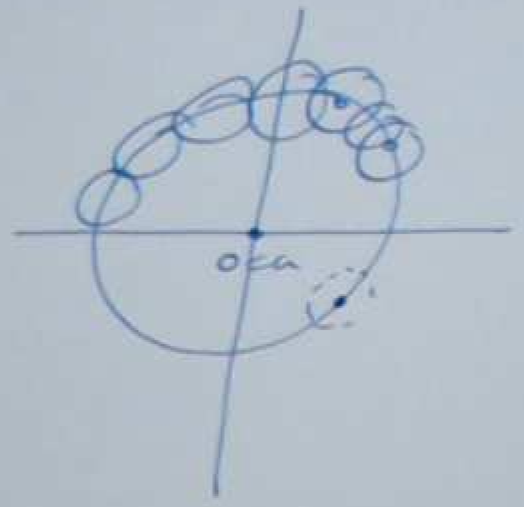
Def^n: If $f(z)$ is analytic in a nbd of w, then w is a regular point for f. Otherwise it is a singular point.

Example $f(z) = \frac{1}{z}$ 0 is singular
all other pts regular.

Thm: Suppose $f(z)$ has a Taylor expansion at a with radius of convergence R, then all points in $|z-a|<R$ is regular, and there is at least one singular point on the circle $|z-a| = R$.

Proof (sketch) we've already shown power series converge in $|z-a|<R$.
suppose every pt on $|z-a|=R$ is regular,
then for each $z \in C$, there is a disc D_z in which f is analytic

D_z cover C [HB] $\Rightarrow$ only need finitely many discs to cover C.



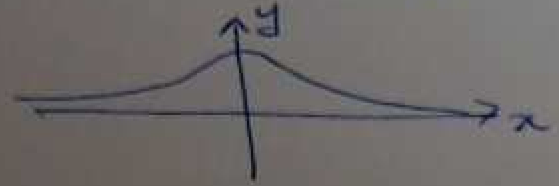
$D_{z_1}, \dots, D_{z_n}$.

let $\epsilon =$ min radius of D_{z_i}

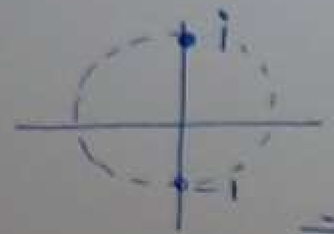
f is analytic in $|z-a| < R + \frac{\epsilon}{2}$.

f has a power series convergent in $R + \frac{\epsilon}{2}$, same as original one $\Rightarrow$ radius of convergence $> R$ ⨳ $\square$.

<u>Example</u> $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$ $R = 1$.



$\mathbb{C}$

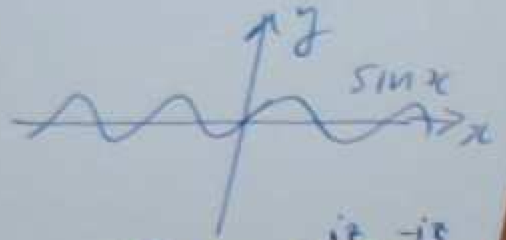


$1+z^2 = (z+i)(z-i)$

sing pts $\pm i$

$\Rightarrow R = 1$.

Thm (Liouville) every bounded entire function is constant. i.e. $f: \mathbb{C} \to \mathbb{C}$ defined on all of $\mathbb{C}$, analytic, there is M s.t. $|f(z)| \le M$ for all $z \in \mathbb{C}$ $\Rightarrow$ $f(z) = c$ (constant).



$\sin z := \frac{e^{iz} + e^{-iz}}{2i}$

Proof $f(z)$ has a Taylor series.

$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$

converges for all $z \in \mathbb{C}$, $|z| < R$ ← for all R.

Cauchy's inequality $(n \geq 1)$: $|c_n| \le \frac{M}{R^n} \to 0$ as $R \to \infty$

$f(z) = c_0$ $\square$.

§10.2 Uniqueness

we have shown that if two power series

$$\left.\begin{array}{l} c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots \\ d_0 + d_1(z-a) + d_2(z-a)^2 + \cdots \end{array}\right\} \text{ same } a!$$

define the same function in an open disc, then $c_n = d_n$ for all n.

replace this with something much weaker.

Thm (uniqueness for power series)

If two power series define the same function on a set of points E, which contains a as a limit point, (a = center) then they have the same coefficients.

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Proof Let z_n be a sequence in E converges to a.

$$\sum_{k=0}^{\infty} a_k (z-a)^k \qquad \sum_{k=0}^{\infty} b_k (z-a)^k \leftarrow \text{two power series.}$$

sum of power series is continuous inside the radius of convergence,

$$a_0 = \lim_{z_n \to a} \sum_{k=0}^{\infty} a_k (z_n - a)^k = \lim_{z_n \to a} \sum_{k=0}^{\infty} b_k (z_n - a)^k = b_0 .$$

$\Rightarrow a_0 = b_0$. (induction on k).

consider. $a_1(z_n - a) + a_2(z_n - a)^2 + \cdots = b_1(z_n - a) + b_2(z_n - a)^2 + \cdots$ on E.

divide by $z_n - a$. $a_1 + a_2(z_n - a)^2 + \cdots = b_1 + b_2(z_n - a)^2 + \cdots$ same on E.

$\Rightarrow b_1 = a_1$, carry on $\square$.

$\Rightarrow b_k = a_k$ for all k.

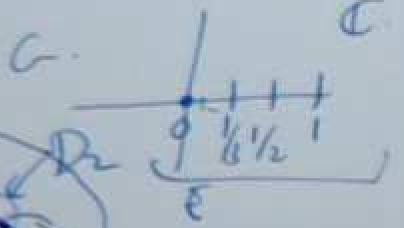
Joseph Maher

Thm (Uniqueness for analytic functions).

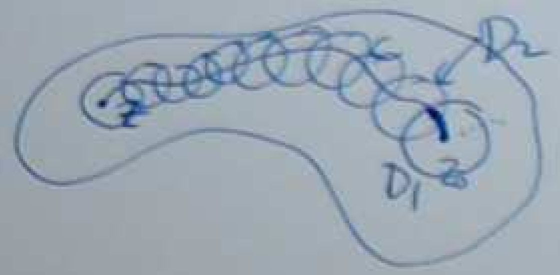
Let f, g be two analytic functions in a domain G

equal $E \subseteq G$, s.t. E has a limit point z_0 in G.

If $f = g$ on E, then $f = g$ in G.



~~Proof~~ Remark: if G = open disc we are done by above.



Proof given $z \in G$, pick a curve C from z_0 to z

[HB] can cover C by finitely many discs $D_1 \dots D_n$

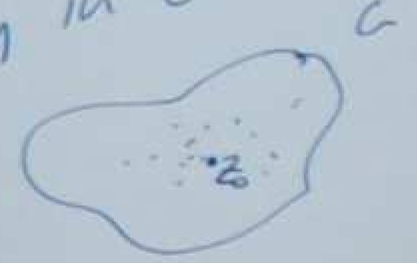
s.t. D_1 contains the limit point z_0, so $f = g$ on D_1

$\Rightarrow f = g$ on $\underbrace{D_1 \cap C}_{\text{arc}}$ which contains a limit point in D_2

so $f = g$ on D_2 and so on... $f = g$ at z $\square$

Thm (Uniqueness for analytic functions)

Let f, g be two analytic functions in a domain G equal $E \subseteq G$, s.t. E has a limit point z_0 in G.



If $f = g$ on E, then $f = g$ in G.

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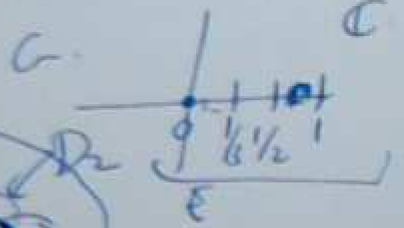

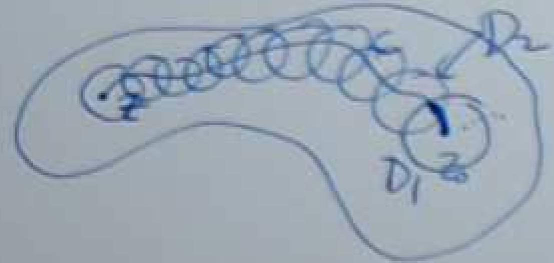

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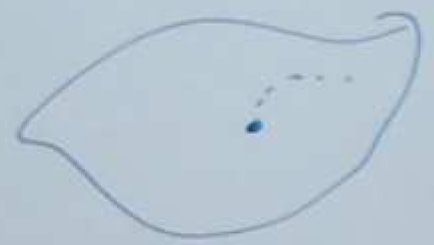
s.t. D_1 contains the limit point z_0, so $f = g$ on D_1

$\Rightarrow$ $f = g$ on $\underbrace{D_1 \cap C}_{\text{arc}}$ which contains a limit point in D_2 so $f = g$ on D_2 and so on... $f = g$ at z $\square$

<u>Remark</u> f analytic and $f(z) = c$ const on any set with a limit point in G, then $f = c$ on G.



let $f(z) \not\equiv 0$ function be analytic in G

and let z_0 be a zero of f, $f(z_0) = 0$.

then f has a power series expansion at z_0.

$$\underbrace{c_0}_{0} + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 + \cdots$$

at least one $c_i \neq 0$ so $f(z) = c_m(z - z_0)^m + c_{m+1}(z - z_0)^{m+1} + \cdots$

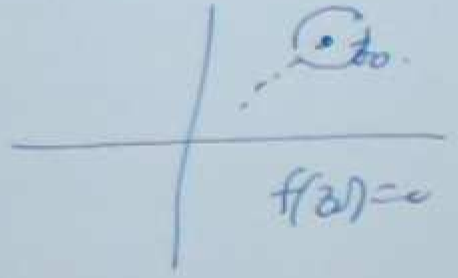
let c_m be the first non-zero one

then we say z_0 is a <u>zero of order m</u>

<u>simple zero</u> if $m = 1$ <u>multiple zero</u> if $m > 1$.

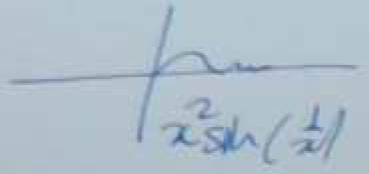
Thm Let f be analytic in a domain G with z_0 a zero of f in G. Then there is a neighborhood of z_0 such that this neighborhood ~~has~~ contains no other zeros.

$\boxed{f(z_0)=0}$



Proof suppose not, then every neighborhood
$|z_0 - z| < \frac{1}{n}$ contains another zero, z_n.

so z_n is a sequence of zeros
limiting to z_0 $\quad f(z_n)=0 \quad f(z_0)=0$



$\Rightarrow f=0$ function # $\square$

Thm Let f be analytic in a domain G with z_0 a zero of f in G. Then there is a neighborhood of z_0 such that this neighborhood ~~has~~ contains no other zeros.

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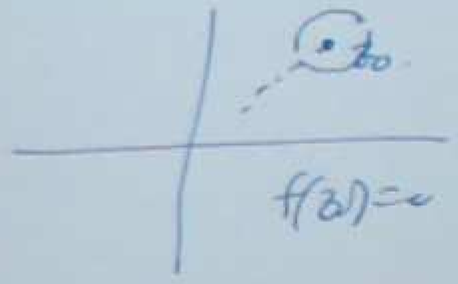

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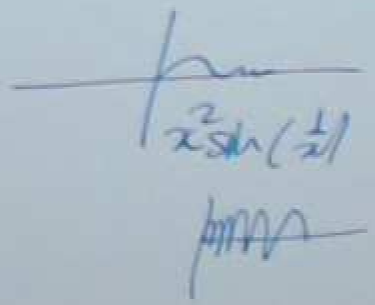


$\Rightarrow f=0$ function # $\square$.

Thm Let f be analytic in a domain G

If $|f(z)|$ is constant in G, then f is constant.

$|f(z)| = \text{const} \Rightarrow f = \text{const}$.

$|f(z)| = 0$ in $G \Rightarrow f(z) = 0$ function in G.

__Proof__ suppose $|f(z)| = 0$ in G

assume $|f(z)| = M > 0$. Let $f(z) = u(x,y) + iv(x,y)$.

$|f(z)|^2 = \{u^2 + v^2\} = M^2$ in G

$\frac{u}{v} = -\frac{v_x}{u_x}$

Differentiate: $2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$, $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$ in G.

$\frac{u}{v} = -\frac{v_y}{u_y}$

both u and v cannot be zero at the same time in G.

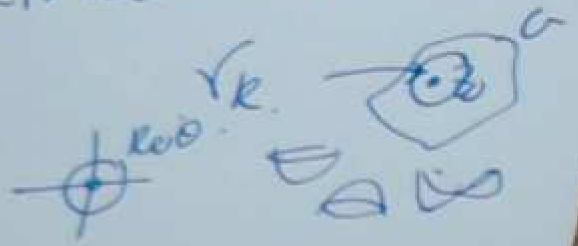
$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$ use CR equations.

$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \Rightarrow \left.\begin{array}{l}\frac{\partial u}{\partial x} = 0\\ \frac{\partial v}{\partial y} = 0\end{array}\right\}$ in G $\Rightarrow$ u & v const

$\Rightarrow f = \text{const}$ $\square$

§10.3 Maximum modulus principle

Thm f analytic, not constant in domain G, then $|f(z)|$ does not have a maximum in G.



Proof suppose, there is a $z_0 \in G$ s.t.

$|f(z_0)| \geq |f(z)|$ for all $z \in G$.

Let γ_R be a small circle of radius R centered at z_0, contained in G.

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{\cancel{Re^{i\theta}}}\,\cancel{i}\cancel{Re^{i\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta$$

if $|f(z_0)| = M$

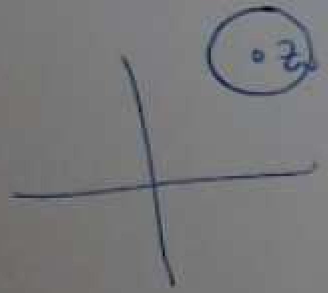
so $M = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{i\theta})|\,d\theta$

$$|f(z_0)| = M = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta \right| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{i\theta})|\, d\theta \qquad (17)$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} M' \, d\theta$$

$$M' = \max_{z \in \gamma_R} |f(z)| \qquad \le \frac{1}{2\pi} 2\pi M_R' = M_R'$$

$M \le M' \Rightarrow M_R' = M$ for all R (sufficiently small).



in fact $|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, dz \right|$

$\Rightarrow |f(z)| = M$ on γ_R.

$|f(z)| = M$ on a small disc around z_0.

by previous $\Rightarrow f(z) = \text{const}$ $\#$ $\square$.

Corollary (Minimum modulus principle).

$f(z)$ analytic in G, not constant, $|f(z)| \neq 0$.
$f(z) \neq 0$ in G.

Then $|f(z)|$ does not have a minimum in G.

Proof $g(z) = \dfrac{1}{f(z)}$ analytic in G, not constant

so $|g(z)|$ has no maximum $\Rightarrow$ $|f(z)|$ has no minimum. $\square$

Corollary Let $f(z)$ be analytic in a bounded domain G, non-constant, cts on ∂G, then there is a point $z_0 \in \partial G$ s.t. $|f(z_0)| = \max_{z \in \underset{\overline{G}}{G \cup \partial G}} |f(z)|$

Proof f cts on $\overline{G}$ closed bounded, $|f(z)|$ cts, $\Rightarrow$ has a max, can't lie in interior of G, so has to lie on ∂G. $\square$

Corollary $f(z)$ analytic, non-constant, non-vanishing in G bounded domain, then there is a $z_1 \in \partial G$ s.t. $|f(z_1)| = \min_{z \in \overline{G}} |f(z)|$.

Proof same as before $\square$.