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$\sum_{n=1}^{\infty} z_n$ $\sum_{n=1}^{\infty} w_n$ $\not\sum_{n=1}^{\infty} z_n w_n$

product

$$(z_1 + z_2 + \cdots)(w_1 + w_2 + \cdots)$$

$$z_1 w_1 + (z_1 w_2 + z_2 w_1) + (z_1 w_3 + z_2 w_2 + z_3 w_1) + \cdots$$

if abs. conv. $s = \sum_{n=1}^{\infty} z_n$ $t = \sum_{n=1}^{\infty} w_n$

product $= st$

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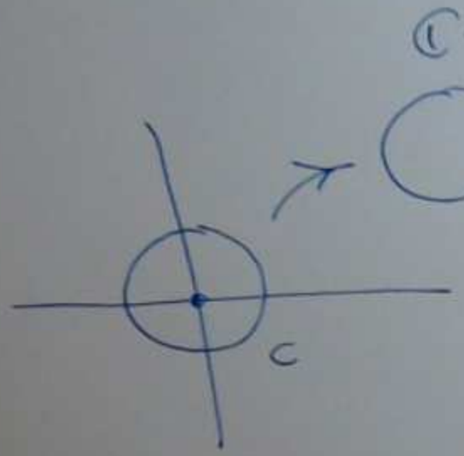
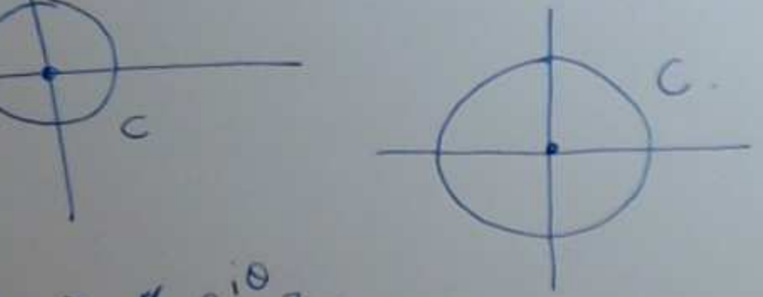
aside.

Möbius maps preserve circles + straight lines.



Q: what about centers?



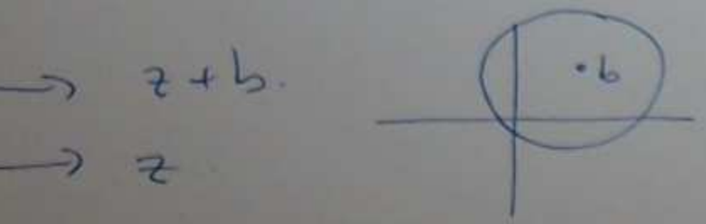
Möbius takes $C \to C$.

where does the center go?

$z \mapsto \alpha e^{i\theta} z$

$z \mapsto \bar{z}$ ← not a Möbius map.

$z \mapsto z + b$.



$z \mapsto z$

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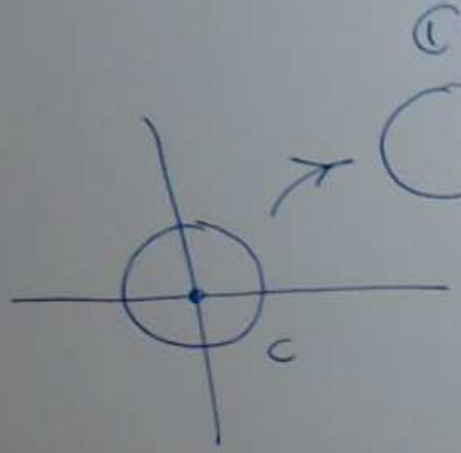
aside.

Möbius maps preserve circles + straight lines.



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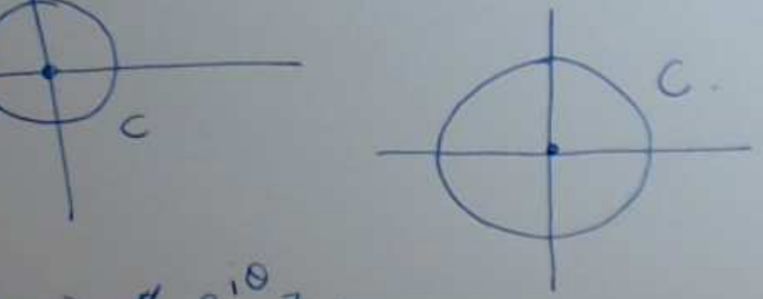

Möbius takes $C \longrightarrow C$.

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$z \longmapsto \bar{z}$ ← not a Möbius map.

$z \longmapsto z + b$.

$z \longmapsto z$.

$z \longmapsto \frac{1}{z}$.

↑ swaps inside and outside.

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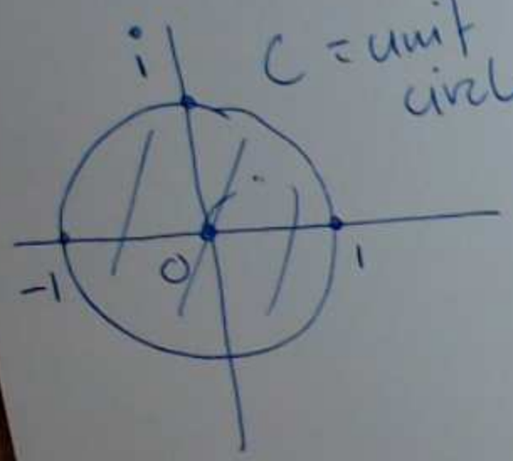


$\mathbb{C}$

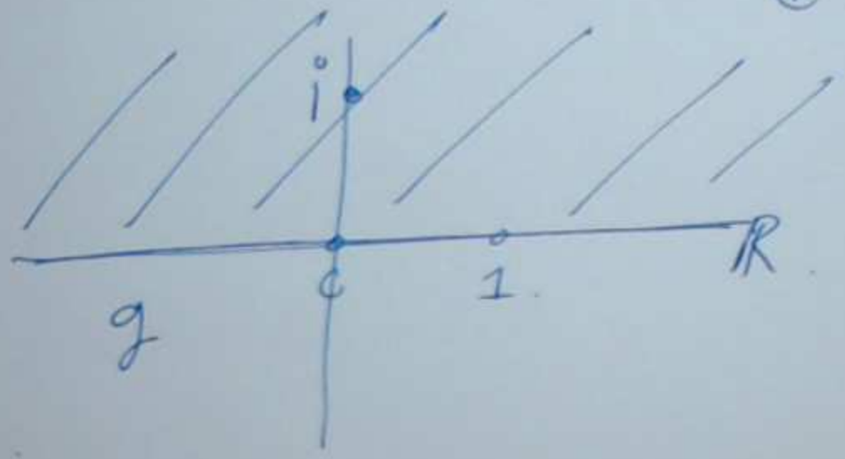
f

f^{-1}

$i \longmapsto 0$

$1 \longmapsto 1$

$-1 \longmapsto \infty$



$\mathbb{C}$ (3)

$$f(z) = \frac{i+1}{i-1}\,\frac{z-1}{z+1}$$

$$f(0) = -\frac{i+1}{(i-1)}\,\frac{(-1-i)}{(-1-i)} = \text{_____}$$

$-1+i$

$z \longmapsto e^{i\theta} z$

$$f(0) = -\frac{1+i}{-1+i}\,\frac{(-1-i)}{(-1-i)} = -\frac{-1-i-i+1}{2} = +i$$

$$\left\{ \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R} \right\} \leftarrow \text{preserves } \mathbb{R}$$

$z \longmapsto z+b \in \mathbb{R}$

$z \longmapsto az \quad a \in \mathbb{R}_{>0}$

$g \in$

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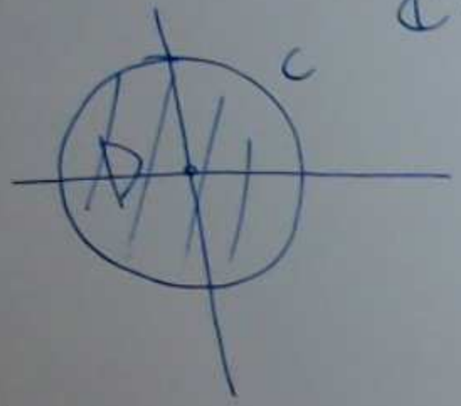
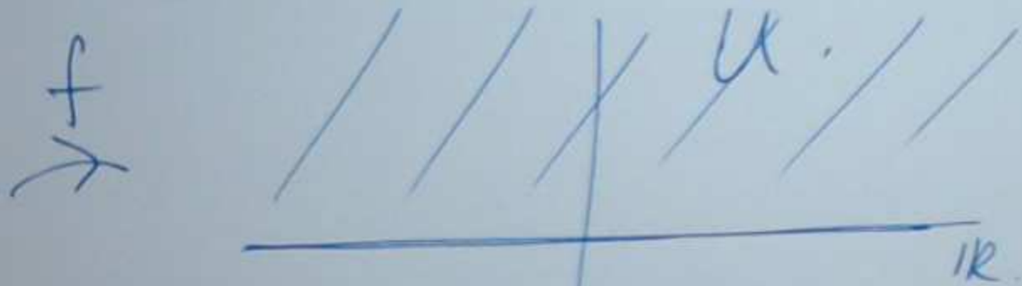


$f(z) = -i\,\frac{z-1}{z+1}$



can take any point in U to any other point

$\Downarrow$

$A = \left\{ \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$ < Möbius group

$\uparrow$ preserve $(\mathbb{R})$

preserve (C) and its interior $=$ $f^{-1} A f$

$\uparrow$ can take any point in D to any other point.

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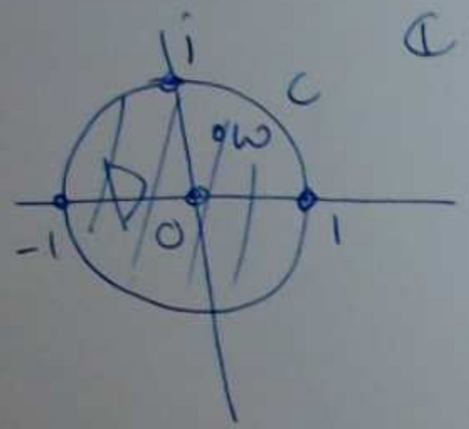
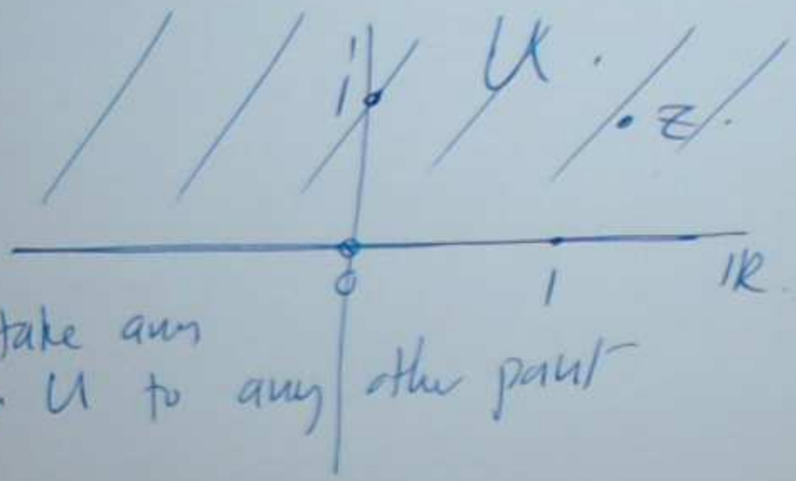


$\mathbb{C}$

f



④

f^{-1} can take any point in U to any other point

$$f(z) = -i\,\frac{z-1}{z+1}$$

$$A = \left\{ \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\} < \text{Möbius group}$$

↑ preserve $(\mathbb{R})$

preserve (C) and its interior. $= f^{-1} A f$

↑ can take any point in D to any other point.

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§9 Multiple valued functions

$\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. (5)

idea —

$$\mathbb{C} \longrightarrow \mathbb{C}$$
$$f: z \longmapsto z^2$$

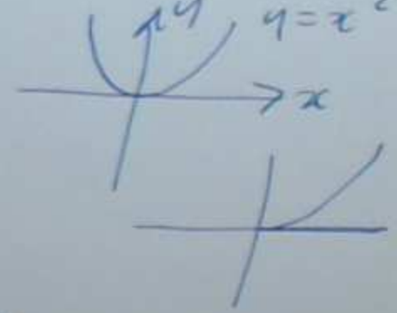
2-1 everywhere except $0, \infty$.

problem f^{-1} not defined.

solution: restrict domain.

(~~bette~~) solution: extend the range.

$$\mathbb{C} \times \mathbb{C} = \mathbb{C}^2 \xleftarrow{f^{-1}} \mathbb{C}$$
$$\mathrm{Sym}_2 \mathbb{C} \qquad \pm\sqrt{z} \longleftarrow z$$
$$(\sqrt{z}, -\sqrt{z}).$$



better solution
~~build~~ covering up.

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Defn $f : G \longrightarrow \mathbb{C}$ is univalent if it is analytic and 1–1.
domain

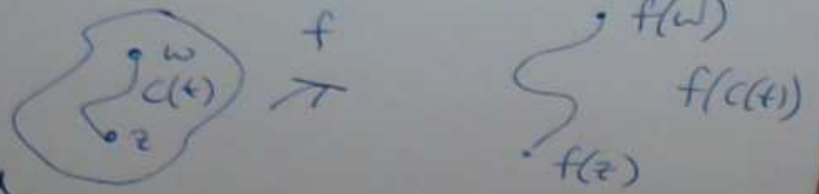
then $f : G \longrightarrow f(G) \subset \mathbb{C}$ is a bijection.

Fact if f is univalent then $f'(z) \neq 0$.

Thm Let $f : G \longrightarrow \mathbb{C}$ be analytic in G and univalent
then $E = f(G)$ is also a domain in G.
↑ open, path connected

Proof . path connected

G is path connected
$\Rightarrow$ $f(G)$ is path connected.



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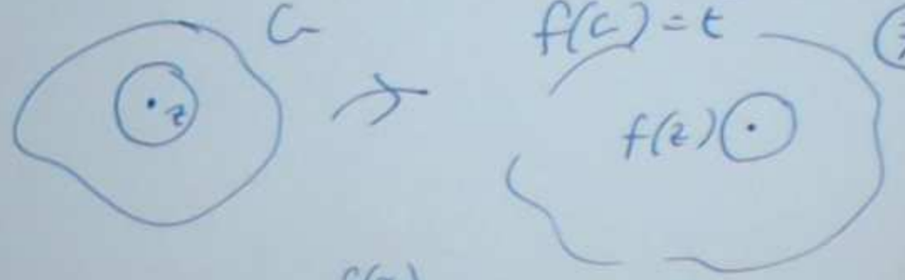
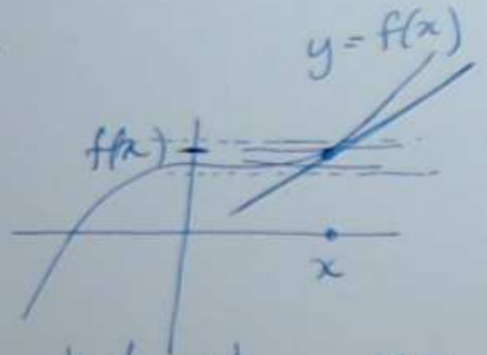
• $E = f(C)$ is open.



this follows from the implicit function theorem.

1d case. $f : \mathbb{R} \longrightarrow \mathbb{R}$. differentiable.



assume $f'(x) \neq 0$

then there is a small open nbd, s.t. $x = f^{-1}(y)$ in that nbd of $f(x)$

$f'(x) \neq 0 \Rightarrow$ local inverse

2d version • need all partial derivatives to exist and be cts

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$(x,y) \mapsto (u(x,y), v(x,y))$

• Jacobian $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$ use Cauchy–Riemann

$= u_x v_y - u_y v_x$

$= (u_x)^2 + (v_x)^2 = |f'(z)|^2$

$\Rightarrow$ locally invertible. $w = f^{-1}(z)$ locally $\square$

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$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

$$= (u_x)^2 + (u_y)^2 \geq 0.$$

$$= |f'(z)|^2 \neq 0.$$

CR $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(not Laplace $u_{xx} + u_{yy} = 0$).

Thm Let $f : G \longrightarrow E$ be an analytic bijection.

let $f^{-1} : E \longrightarrow G$. Then f^{-1} is univalent

and $(f^{-1}(\omega))' = \frac{1}{f'(z)}$ where $\omega = f(z)$.

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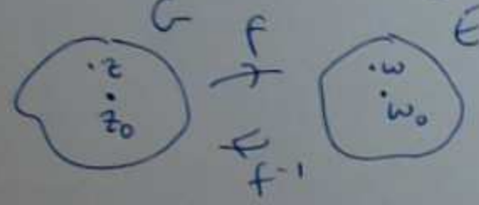
Proof • f^{-1} exists as f is a bijection.

1-1

• f^{-1} is cts by 2d implicit ~~value~~ theorem. functions are 1 to 1 and onto.



$f(z_0) = w_0$ $\quad z_0 = f^{-1}(w_0)$

$f(z) = w$ $\quad z = f^{-1}(w)$

$$f^{-1}(w_0) = \lim_{w \to w_0} \frac{f^{-1}(w_0) - f^{-1}(w)}{w_0 - w} = \lim_{w \to w_0} \frac{z_0 - z}{w_0 - w}$$

f, f^{-1} cts so $w \to w_0$ iff $z \to z_0$

$$= \lim_{z \to z_0} \frac{1}{\frac{w_0 - w}{z_0 - z}} = \frac{1}{\lim_{z \to z_0} \frac{f(z_0) - f(z)}{z_0 - z}} = \frac{1}{f'(z_0)} \quad \square$$

(z_0 above $f(z_0)$, z above $f(z)$)

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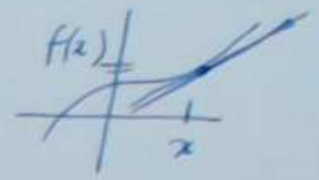
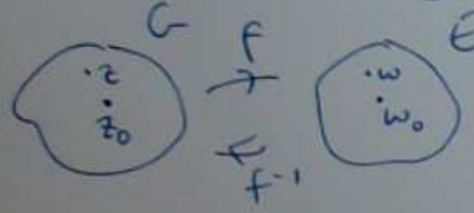
Proof • f^{-1} exists as f is a bijection.



1-1

• f^{-1} is cts by 2d implicit function theorem. ~~theorem~~ one to one and onto.



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(z_0, z)

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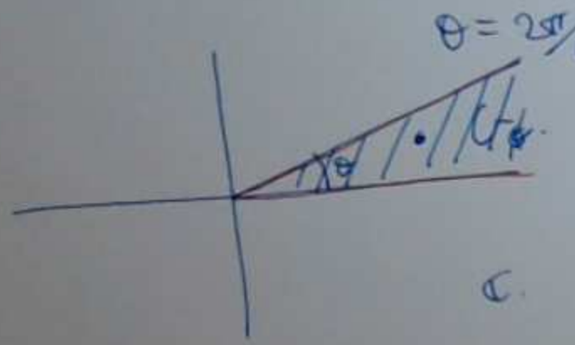
Examples

$f : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto z^n$

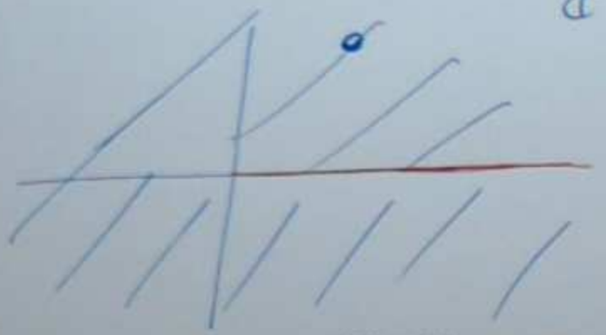
$n-1$ function except at $0, \infty$.

- restrict domain: f is univalent on $0 < \arg z < \frac{2\pi}{n}$.



f

$z \mapsto z^n$



define

$f^{-1} : E \longrightarrow \mathbb{C}$

$z \longmapsto \sqrt[n]{z}$

$E = \mathbb{C} \setminus$ non-neg ~~+ve~~ real axis

$\mathbb{C} \setminus [0, \infty) \subseteq \mathbb{R}$

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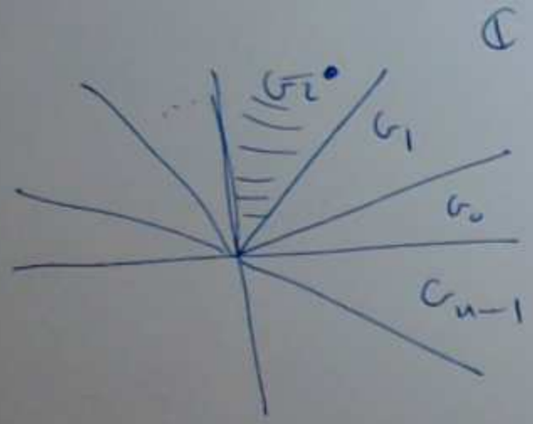
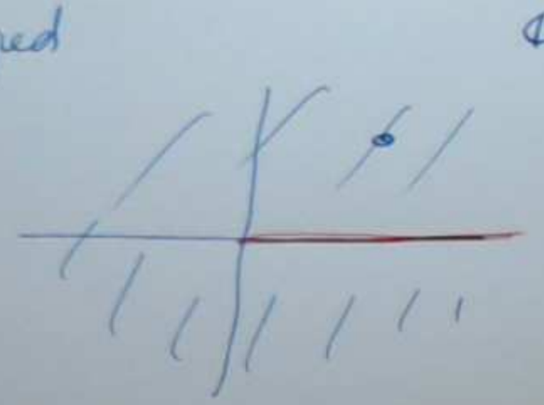
§9.2 Branches and branch points.

$f: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^n$ consider $\frac{2\pi k}{n} < \arg z < \frac{2\pi(k+1)}{n}$.



each σ_i gets mapped to $\mathbb{C}$

f



$f|_{\sigma_k}: \sigma_k \longrightarrow \mathbb{C}$ is a bijection

can define an inverse $(\sqrt[n]{z^{-1}})_k : \mathbb{C} \longrightarrow \sigma_k$

could define $\mathbb{C} \longrightarrow \mathbb{C}^n$

$z \longmapsto ((\sqrt[n]{z^{-1}})_1, (\sqrt[n]{z^{-1}})_2, \dots (\sqrt[n]{z^{-1}})_n)$.

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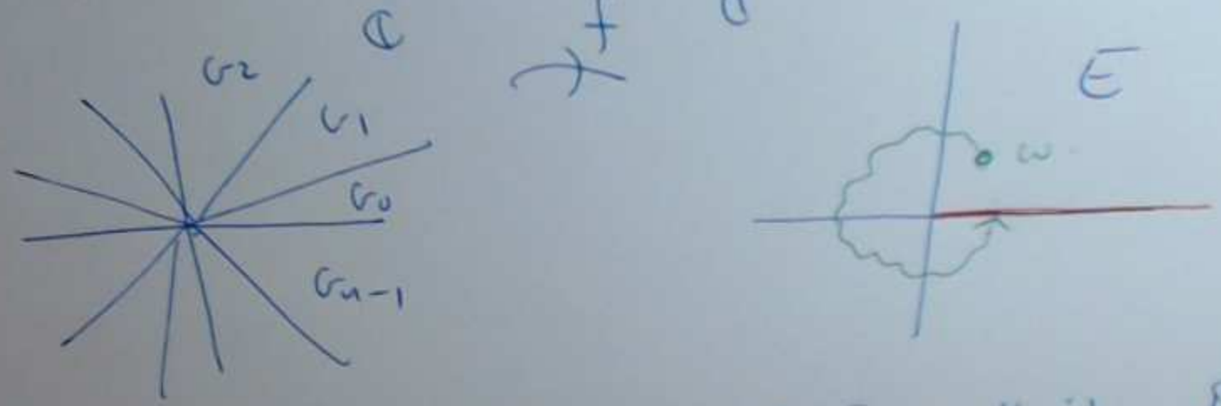
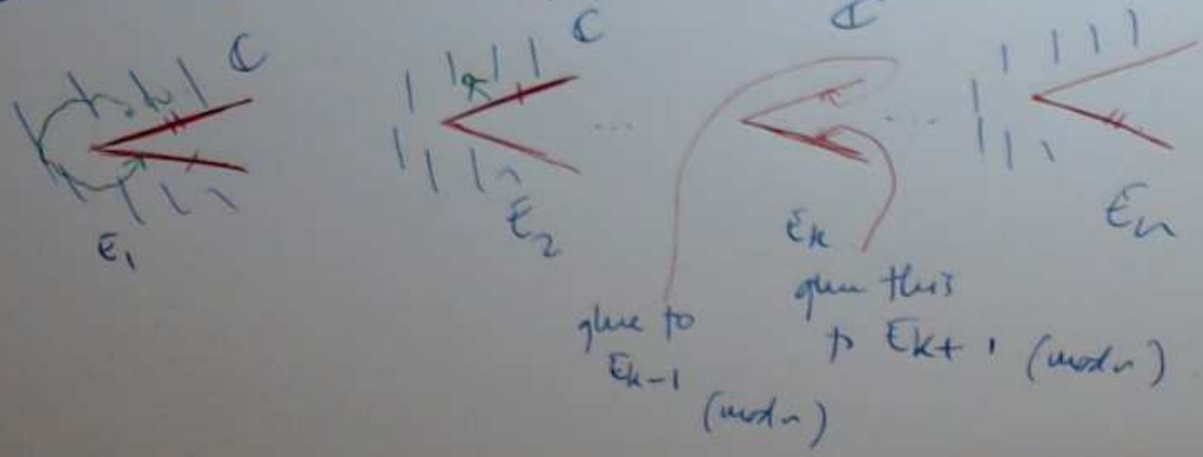
branch | Riemann surface | extend the range.



for each G_k take a copy of $\mathbb{C}$, call it E_k.



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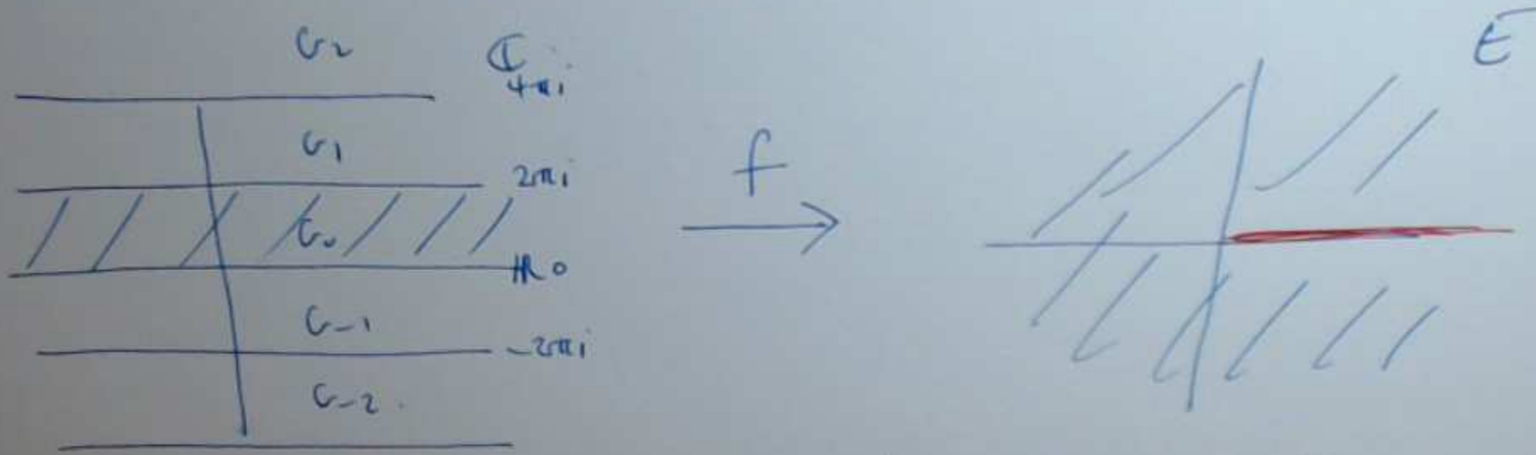
Example $f : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto e^z = e^x e^{iy}$



$f|_{G_0} : G_0 \longrightarrow \bar{\mathbb{C}}$

bijection

G_k $\quad 2\pi k < \mathrm{Im}(z) < 2\pi(k+1)$.

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restricting domains: $\sigma_k \longrightarrow E$ bijection

$(\log(z))_k : E \longrightarrow \sigma_k$.

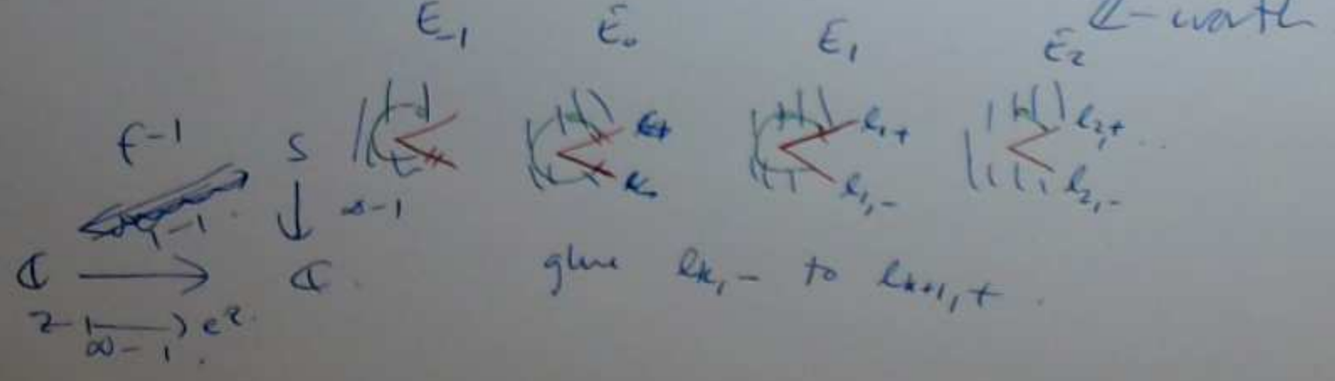
multivalued log: $\mathbb{C} \longrightarrow \mathbb{C}^{\mathbb{Z}}$

$z \longmapsto (\ldots, (\log z)_k, \ldots)$

$\uparrow \sigma_k$

branch / Riemann surface $S \leftarrow$ build this out of E_k's.

$\mathbb{Z}$-worth



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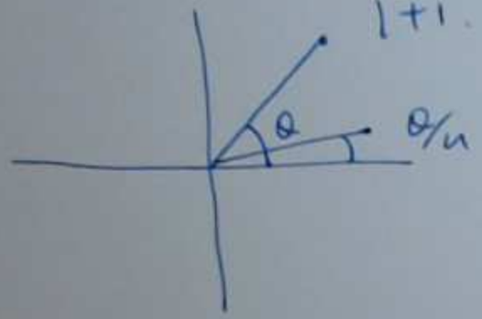
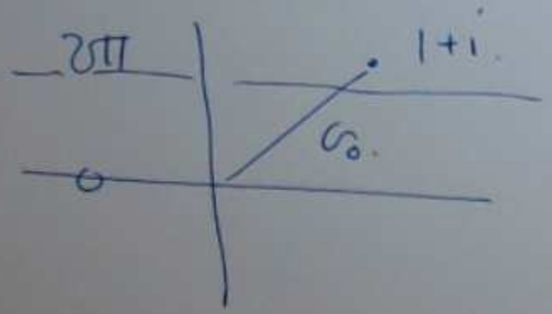
recall



$\rightsquigarrow \sqrt[n]{1+i}$



$\log(1+i)$

$z = x+iy \quad e^z = e^x e^{iy}$

$|e^z| = |e^x||e^{iy}|$

$|e^z| = |e^x|$

$$\log(z) = \log|z| + i\arg(z) + i2\pi k \qquad k \in \mathbb{Z}$$

$$\log(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4} + i2\pi k$$

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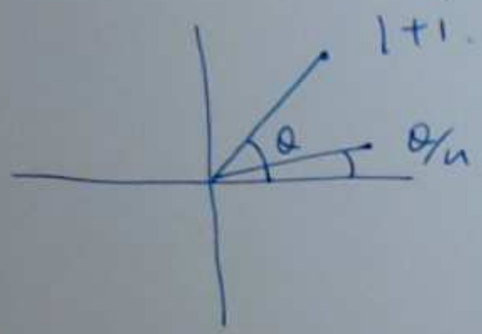
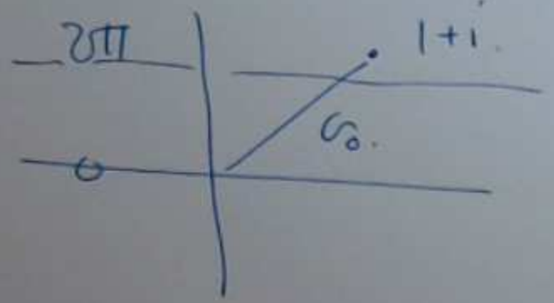
recall



$\rightsquigarrow \quad \sqrt[n]{1+i}$

$\log(1+i)$



$z = x+iy \quad e^z = e^x e^{iy}$

$|e^z| = |e^x||e^{iy}|$

$|e^z| = |e^x|$

$\log(z) = \log|z| + i\arg(z) + i2\pi k \qquad k \in \mathbb{Z}$

$\log(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4} + i2\pi k$

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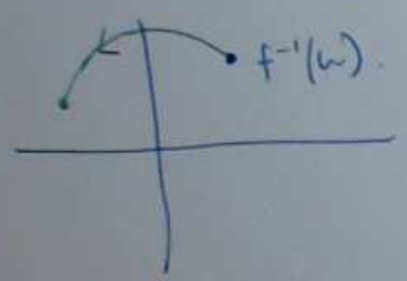

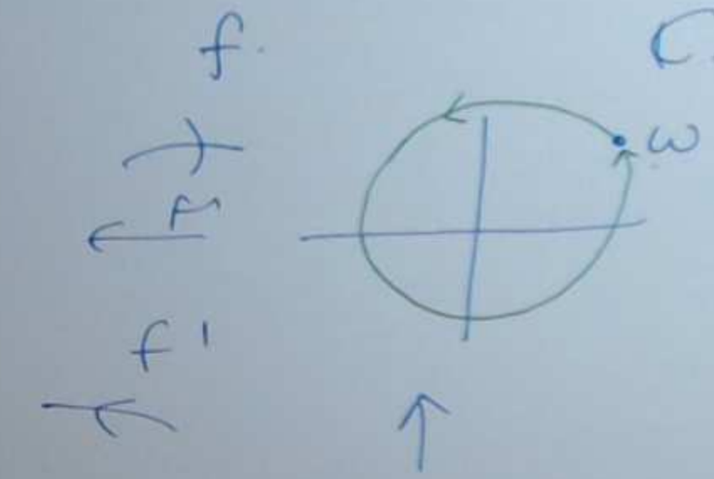

next time

$z \mapsto z^2$

$z \mapsto z^2 - 2z$

$z-1$ ↓

$\mathbb{C} \to \mathbb{C}$

ex $z \mapsto z^n$

$z \mapsto e^z$

branch pts are $0, \infty$.

branch pt vs regular point

a point $w \in S$ is a branch point if a small loop around w changes the branch

(in these examples Ek's are the branches)

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Joseph Maher