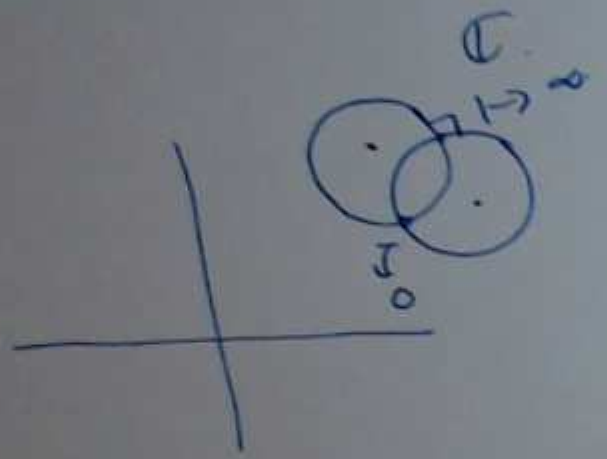
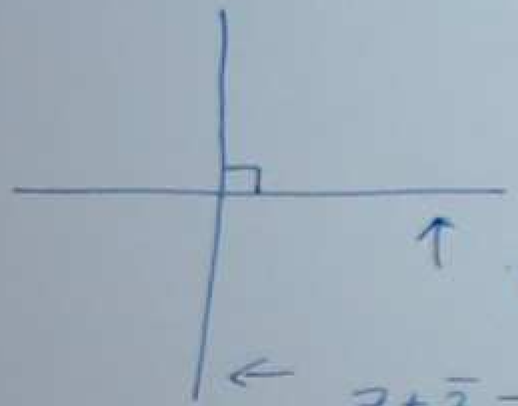


$\xrightarrow{f}$

$\xleftarrow{f}$



$iz - i\bar{z} = 0$

$B_2 = i$

$A_2 = 0$

$C_2 = 0$

$z + \bar{z} = 0$

$B_1 = 1 \quad A_1 = 0$

$C_1 = 0$

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<u>Möbius maps</u> $f(z) = \dfrac{az+b}{cz+d}$ $\quad f : \mathbb{C} \longrightarrow \mathbb{C}$.

$ad - bc \neq 0$. $\quad f : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$.

<u>Th^m</u> Möbius maps preserve {circles and straight lines}.

<u>Prove</u> <u>recall</u>: general equation of a circle / straight line is $A|z|^2 + Bz + \bar{B}\bar{z} + C = 0$

<u>special cases</u> $\quad A, C \in \mathbb{R}$

$A|az|^2 + Baz + \bar{B}\bar{a}\bar{z} + C$. $\quad B \in \mathbb{C}$

① $f(z) = az$ $\quad A|a|^2|z|^2 + (Ba)z + \overline{Ba}\bar{z} + C$ ✓

$a \in \mathbb{C}$
$a \neq 0$

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(3)

(2) $z \mapsto z+b$

$$A|z|^2 + Bz + \overline{B}\overline{z} + C = 0$$

$f(z) = z+b$

$z = f(z) - b$

$$A\,\underbrace{|f(z)-b|^2}_{= (f(z)-b)\overline{(f(z)-b)}} + B(f(z)-b) + \overline{B}\left(\overline{f(z)-b}\right) + C$$

$$A|f(z)|^2 - A f(z)\overline{b} - A\overline{f(z)}b + \underbrace{b\overline{b}}_{|b|^2} + Bf(z) - Bb + \overline{B}\,\overline{f(z)} - \overline{B}b + C$$

$$A|f(z)|^2 + f(z)[B - A\overline{b}] + \overline{f(z)}[\overline{B} - Ab] + \underbrace{C - Bb + b\overline{b} - \overline{B}b}_{\text{real}}$$

conjugates

$B' = B - A\overline{b}$

$C' = \sim$

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3 $f: z \longmapsto \frac{1}{z}$.

$f(z) = \frac{1}{z}$.

$z = \frac{1}{f(z)}$.

$$A|z|^2 + Bz + \bar{B}\bar{z} + C = 0.$$

$$A\left|\frac{1}{f(z)}\right|^2 + B\overline{\frac{1}{f(z)}} + \bar{B}\frac{1}{f(z)} + C = 0$$

$$A\frac{1}{f(z)\overline{f(z)}} + \frac{B}{\overline{f(z)}} + \frac{\bar{B}}{f(z)} + C = 0$$

$$\underbrace{A}_{\text{real}} + B\overline{f(z)} + \bar{B}f(z) + \underbrace{C|f(z)|^2}_{\text{real}} = 0$$

$B' = \bar{B}$

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<u>finally</u> $f(z) = \dfrac{az+b}{cz+d}$ is a composition of maps of the form

$$\left.\begin{array}{l} z \longmapsto az \\ z \longmapsto z+b \\ z \longmapsto \frac{1}{z} \end{array}\right\} \ (*)$$

do long division.

$$\begin{array}{r|l} & \;\; a/c \\ \hline cz+d & \;\; az+b \\ & \;\; az + \frac{ad}{c} \\ \hline & \;\; b - \frac{ad}{c} \end{array}$$

$$f(z) = \frac{a}{c} + \frac{b - ad/c}{cz+d}.$$

this is a composition of $(*)$ $\square$

Thm Given three (ordered, distinct) points a, b, c there is a unique Möbius map taking them to $0, 1, \infty$.

Proof $a \longmapsto 0$
$b \longmapsto 1$
$c \longmapsto \infty$

$$f(z) = \frac{b-c}{b-a} \, \frac{z-a}{z-c}$$

existence ✓

uniqueness: if $a \mapsto 0$ numerator has a factor of $\frac{z-a}{z-c}$
$c \mapsto \infty$ denominator
$b \mapsto 1$ this fixes the ratio of $\frac{b-a}{b-c}$ ✓

□

<u>Warning</u> $f(z)=\dfrac{az+b}{cz+d}$ but different a,b,c,d can give same map.

Consider $\lambda a, \lambda b, \lambda c, \lambda d$. $\lambda \neq 0$.

$$f(z)=\frac{\lambda a z+\lambda b}{\lambda c z+\lambda d}=\frac{az+b}{cz+d}.$$

<u>Def^{n}</u> cross ratio of x,y,z,t is $\dfrac{z-x}{z-y}\cdot\dfrac{t-x}{t-y}$

<u>Th^{m}</u> cross ratio is preserved by Möbius maps.

<u>check</u> ① $z \xrightarrow{f} az$ $\qquad \dfrac{az-ax}{az-ay}\cdot\dfrac{at-ax}{at-ay}=\dfrac{z-x}{z-y}\cdot\dfrac{t-x}{t-y}$ ✓

x,y,z,t

$\to f(x), f(y), f(z), f(t)$

2 $f: z \longmapsto z+b$.

$$\frac{z-x}{z-y} \cdot \frac{t-\cancel{x}\,y}{t-\cancel{y}\,x}$$

$a, b, c, d \mapsto f(a), f(b), f(c), f(d)$

$$\frac{z+b-(x+b)}{(z+b)-(y+b)} \, \frac{(t+b)-(x+b)}{(t+b)-(y+b)} = \frac{z-x}{z-y} \cdot \frac{t-\cancel{x}\,y}{t-\cancel{y}\,x}$$

3 $f: z \longmapsto \frac{1}{z}$.

$$\frac{\frac{1}{z}-\frac{1}{x}}{\frac{1}{z}-\frac{1}{y}} \cdot \frac{\frac{1}{t}-\frac{1}{\cancel{x}\,y}}{\frac{1}{t}-\frac{1}{\cancel{y}\,x}} = \frac{\frac{x-z}{xz} \quad \frac{y-t}{ty}}{\frac{y-z}{yz} \quad \frac{x-t}{xt}}$$

$$= \frac{x-z}{y-z} \cdot \frac{y-t}{x-t} \quad \square.$$

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(10)

cross ratio: x, y, z, t

$$\cancel{\frac{y-z}{y-x}\ \frac{t-x}{t-z}}$$

$z \longmapsto az$

$z \longmapsto z+b$

$z \longmapsto \frac{1}{z}$.

$$\boxed{(z_1\ z_2\ z_3\ z_4) = \frac{z_3-z_1}{z_3-z_2}\ \frac{z_4-z_2}{z_4-z_1}}$$

$$x, y, z, t = \frac{z-x}{z-y}\ \frac{t-y}{t-x}$$

$(0, 1, \infty, \infty) \mapsto \infty$

cross ratio

$(z_1, z_2, z_3, z_4) \mapsto \left(\infty, 0, 1, \frac{z_3-z_1}{z_3-z_2}\ \frac{z_4-z_2}{z_4-z_1}\right)$

$(z_3, z_1, z_2, z_4) \mapsto (\infty, 0, 1, \text{cross-ratio})$.

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<u>Corollary</u> Let C be a circle determined by $z_1, z_2, z_3 \in \mathbb{C}$.

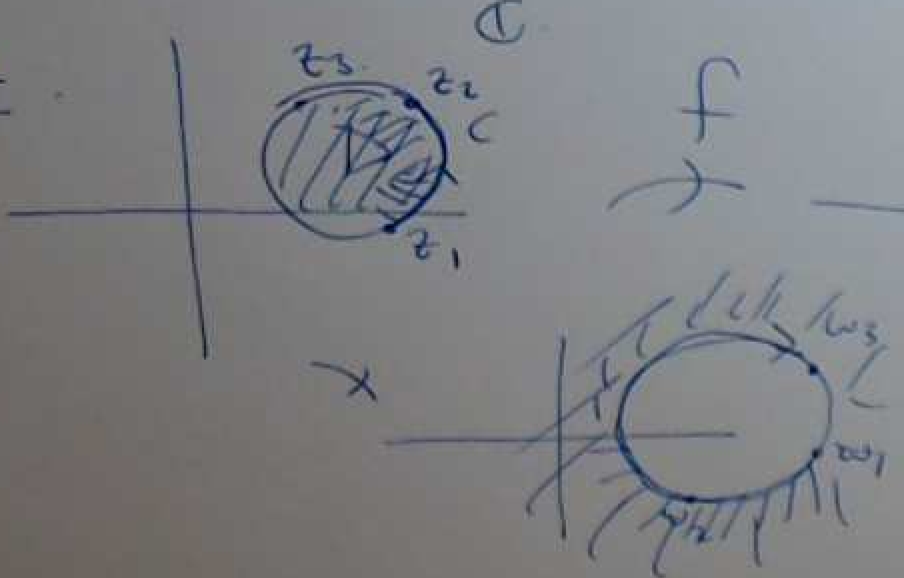
C' $w_1, w_2, w_3 \in \mathbb{C}$.

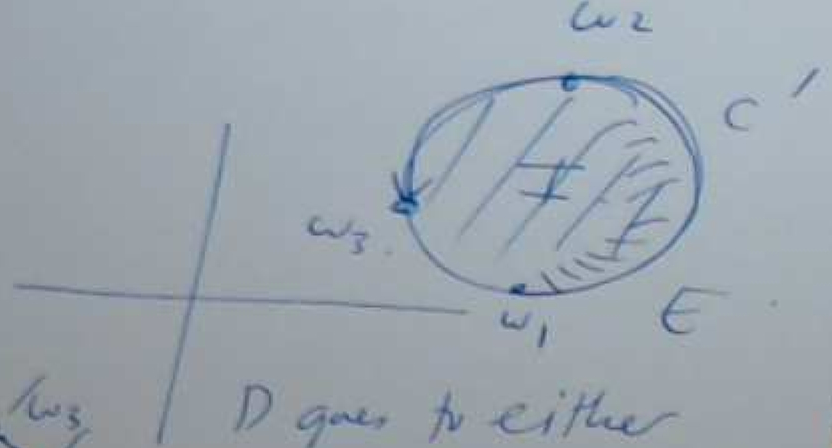
then there is a unique Möbius map taking

$$(z_1, z_2, z_3) \longmapsto (w_1, w_2, w_3).$$

$$C \longmapsto C'. \qquad \square$$

<u>Remark</u>.

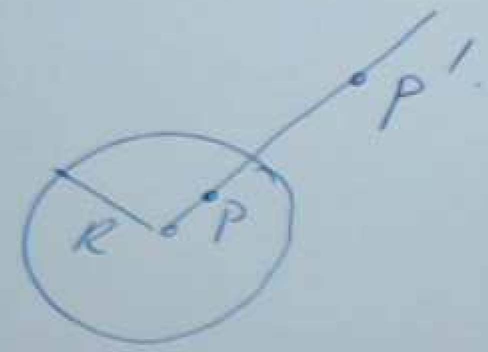




D goes to either I or E

Q: which one?

__Recall__ P, P' are __symmetric__ wrt a circle C if they lie on the same ray through the center and $OP \cdot OP' = (\text{radius})^2$
R^2.



__Thm__ Two points P, P' are symmetric wrt C iff every straight line and circle through P and P' is orthogonal to C

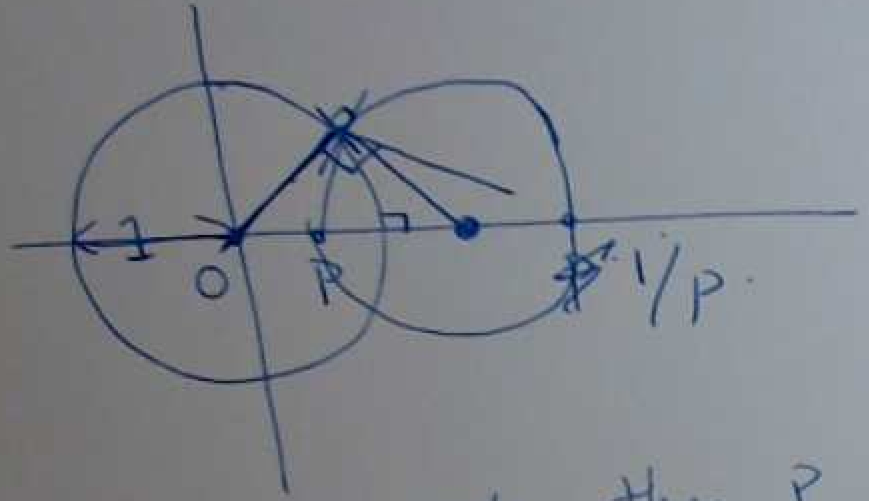
__Proof__ $(\Rightarrow)$ wlog center $0 \in \mathbb{C}$, $R = 1$.
rotate so P, P' lie on +ve x-axis.

~~OP~~ $P, P' \in \mathbb{R}$

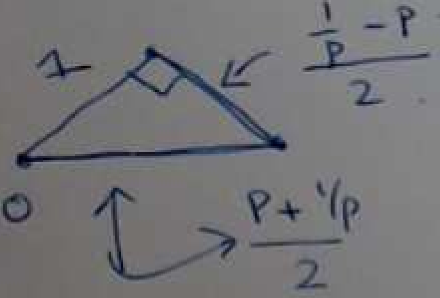
$OP \cdot OP' = 1$

$PP' = 1$

straight line thru P, P' ✓.



consider a circle thru P, P'



$$1^2 + \left(\frac{\frac{1}{p} - p}{2}\right)^2$$

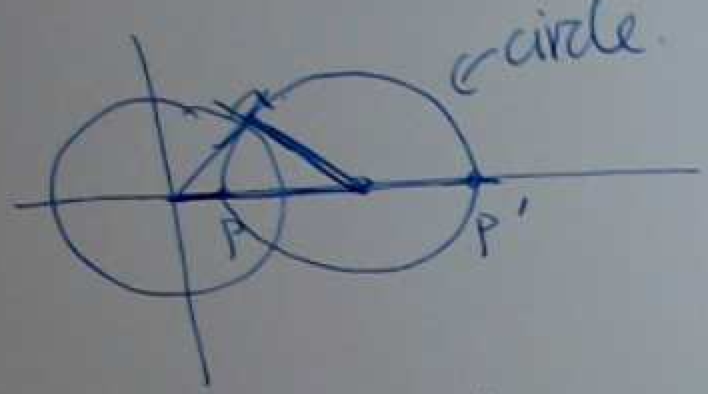
$$\frac{4}{4} + \frac{\frac{1}{p^2} - 2 + p^2}{4} = \frac{\frac{1}{p^2} + 2 + p^2}{4}$$

$$= \left(\frac{\frac{1}{p} + p}{2}\right)^2$$

satisfies pythagoras

$\Rightarrow$ right angled

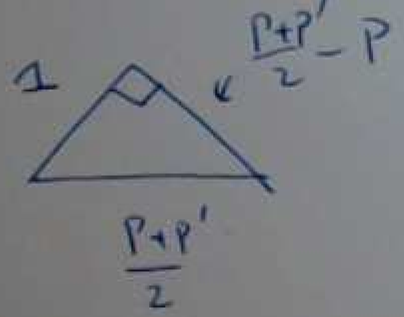
$\Rightarrow$ suppose every circle straight line orthogonal.



tangents $\perp$

$\Rightarrow$ triangle is right triangle.

by Pythagoras.



$$1^2 + \left(\frac{p+p'}{2} - p\right)^2 = \left(\frac{p+p'}{2}\right)^2$$

$$1^2 + \left(\frac{p'-p}{2}\right)^2 = \frac{p^2 + 2pp' + p'^2}{4}$$

$\Rightarrow$

$$\frac{4}{4} + \frac{\cancel{p'^2} - 2pp' + \cancel{p^2}}{4} = \frac{\cancel{p^2} + 2pp' + \cancel{p'^2}}{4}$$

$$1 = pp'$$

$\Rightarrow$ p, p' are symmetric $\square$.

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Thm let z_1 and z_2 be symmetric wrt C

let f be a Möbius transformation,

then $f(z_1)$ and $f(z_2)$ are symmetric wrt $f(C)$

Proof f takes circles/straight lines to circles and straight lines.

preserves angles,

so it takes orthogonal circles and straight lines to orthogonal circles and straight lines.

$\Rightarrow$ Möbius maps take symmetric pts wrt C to symmetric pts wrt $f(C)$ $\square$

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$$\alpha(z) = \frac{az+b}{cz+d} \qquad \beta(z) = \frac{ez+f}{gz+h}$$

$$\beta(\alpha(z)) \quad \frac{e\,\frac{az+b}{cz+d} + f}{g\,\frac{az+b}{cz+d} + h} = \frac{eaz + eb + f(cz+d)}{gaz + gb + h(cz+d)}$$

$$= \frac{(ea+fc)z + (eb+fd)}{(ag+ch)z + (gb+hd)}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{az+b}{cz+d}$$

$ad - bc \neq 0$

$\det \neq 0$

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + hd \end{bmatrix}$$

Thm there is a homomorphism $GL(2,\mathbb{C}) \xrightarrow{\phi}$ Möbius maps

$\phi(AB) = \phi(A)\phi(B)$ $\quad A, B$

$\Phi: \quad GL(2,\mathbb{C}) \longrightarrow$ Möbius maps.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{az+b}{cz+d}.$$

not injective / 1-1

$$\begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} \longmapsto \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{az+b}{cz+d}.$$

$\lambda \in \mathbb{C} \setminus \{0\}$

homomorphism

ker $\quad \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$SL(2,\mathbb{C}) \longrightarrow$ Möbius maps

~~kernel~~.

$\cancel{\pm I}$

$\cancel{\pm I}$

Φ: kernel?

$SL_2\mathbb{R} \longrightarrow$ Möbius real coeffs

$\pm I \longrightarrow z \mapsto z$

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$$f(z) = \frac{az+b}{cz+d}.$$

$$z \mapsto 2z.$$

$$z \mapsto \frac{1}{2}z.$$

$$f^{-1}(z) = \cancel{\frac{1}{ad-bc}} \frac{dz-b}{-cz+a}$$

$$\frac{2z+0}{0+1}.$$

$$\frac{z+0}{0+2}.$$

$$z \mapsto \frac{1}{2}$$