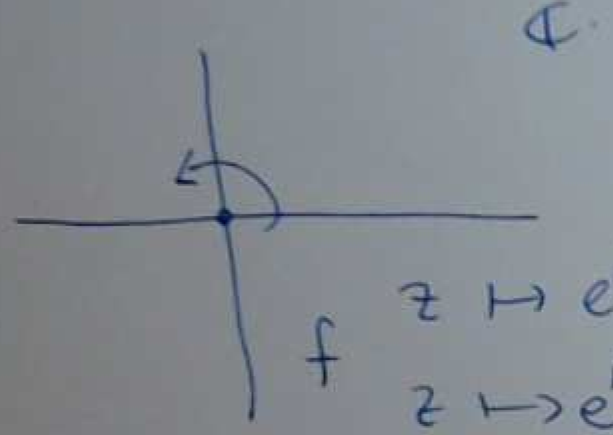
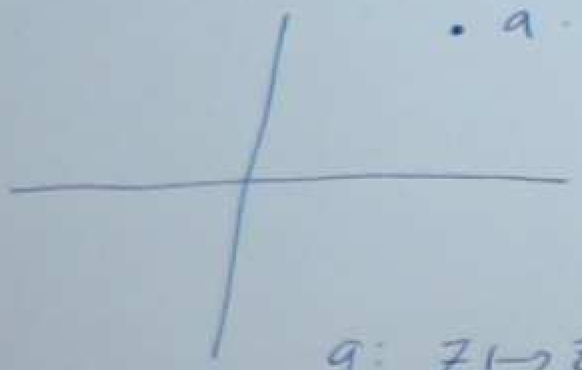
①

half turns





f $\quad z \mapsto e^{i\theta} z$

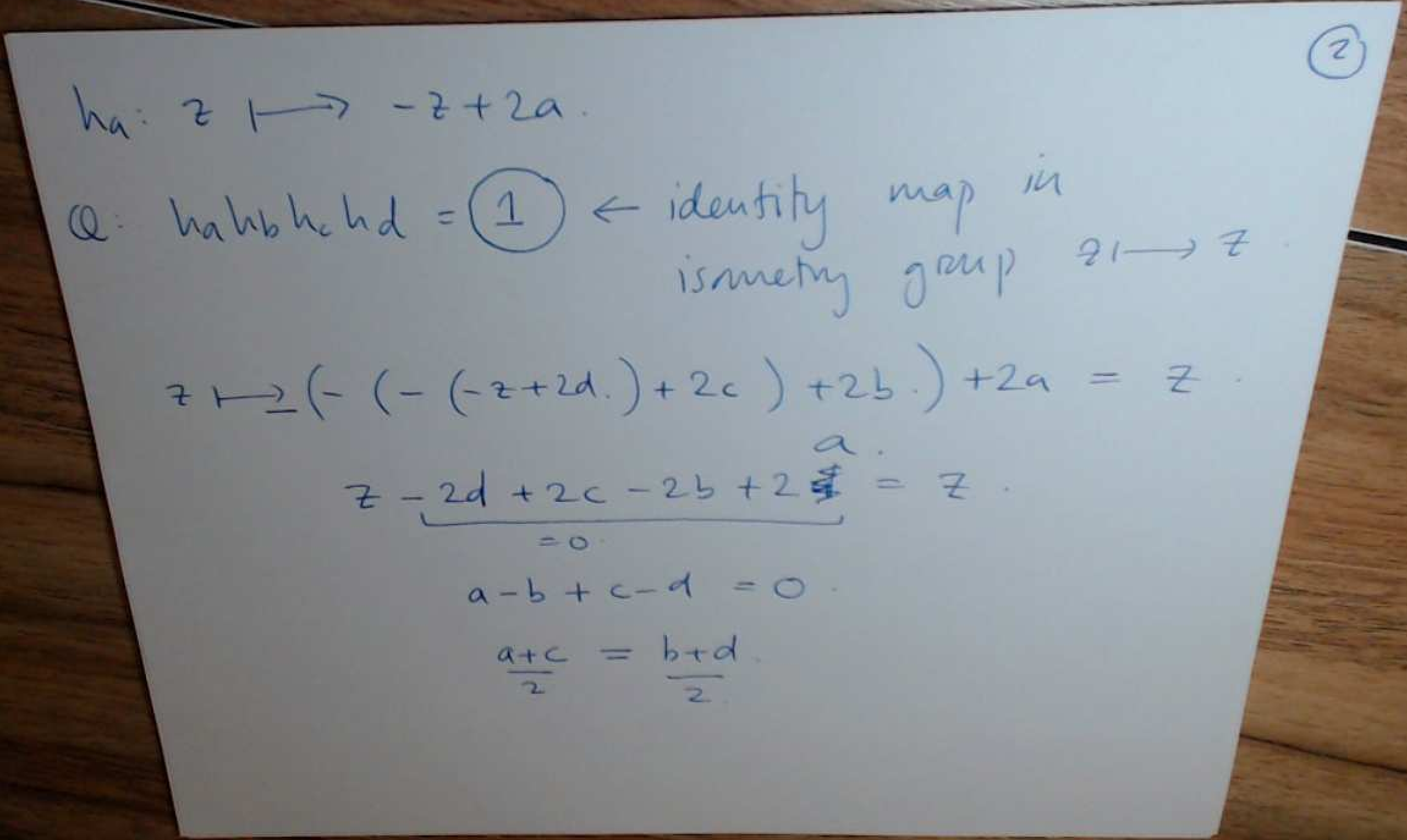
$z \mapsto e^{i\pi} z$

$z \mapsto -z$

$g: z \mapsto z+a$

$gfg^{-1}: z \longmapsto z-a \longmapsto -(z-a) \longmapsto -(z-a)+a$

$\overset{\shortparallel}{h_a}: z \longmapsto -z+2a$



$h_a : z \longmapsto -z + 2a.$

Q: $h_a h_b h_c h_d = \boxed{1}$ ← identity map in isometry group $z \longmapsto z.$

$$z \longmapsto -(-(-(-z+2d)+2c)+2b)+2a = z.$$

$$z \underbrace{-2d+2c-2b+2\overset{a}{\cancel{z}}}_{=0} = z.$$

$$a-b+c-d = 0.$$

$$\frac{a+c}{2} = \frac{b+d}{2}.$$

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③

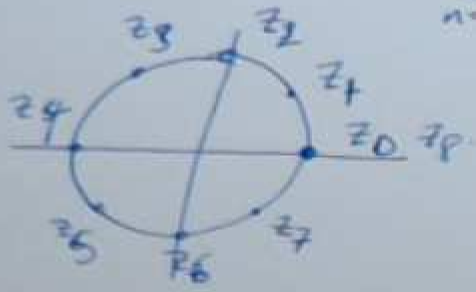
§6 Q9 a) $\sum_{n=0}^{1} \frac{1}{(1+i)^n} \leftarrow$ geometric series.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

$$\frac{1}{1-\frac{1}{1+i}}.$$

$|r|<1$.

b) $\sum_{n=0}^{\infty} \cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}$ $\qquad \sum_{n=0}^{\infty} e^{i\frac{n\pi}{4}}$ $\qquad \sum_{n=0}^{\infty} z_n$



$$s_n = \underbrace{z_0 + z_1 + z_2 + z_3 + z_4}\ + z_5 \qquad + z_n$$

$s_7 = 0$

$s_0 = 1$

$s_1 = 1 + \frac{1+i}{\sqrt{2}}$

$s_1 = 1 + \frac{1+i}{\sqrt{2}} + i$

$s_7 = 0$

$(s_n) \leftarrow$ does not converge

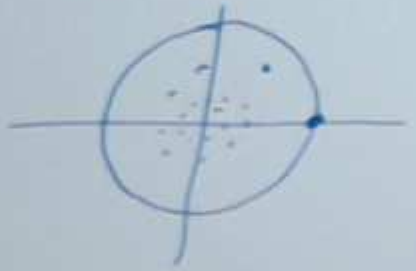
(oscillating) $\qquad$ <u>divergent</u>

Joseph Maher

c) $\sum_{n=0}^{\infty} \left| \frac{1}{(1+i)^n} e^{i\pi n/4} \right|$

$\sum_{n=0}^{\infty} z_n$

$s_n = z_0 + z_1 + z_2 + \cdots + z_n$



$\sum_{n=0}^{\infty} \frac{1}{\sqrt{2}^n} \leftarrow$ geometric series

converges

$\Rightarrow \sum z_n$ absolutely convergent

so converges.

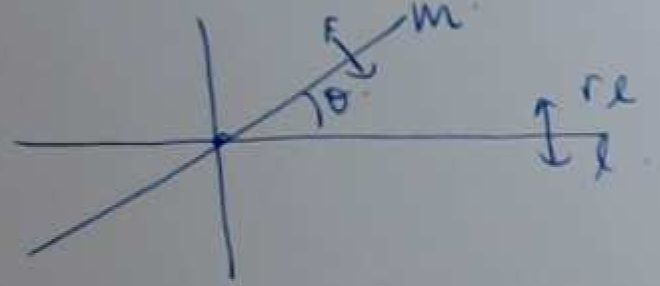
$r_\ell r_m = r_m \& r_\ell$

- without loss of generality
- explicit forms for isometries

- wlog $\ell =$ x-axis.

- two cases: m parallel

  ⊛ m intersects.



$r_\ell \quad z \mapsto \bar{z}$

$\rho : z \mapsto e^{i\theta} z$ rotation

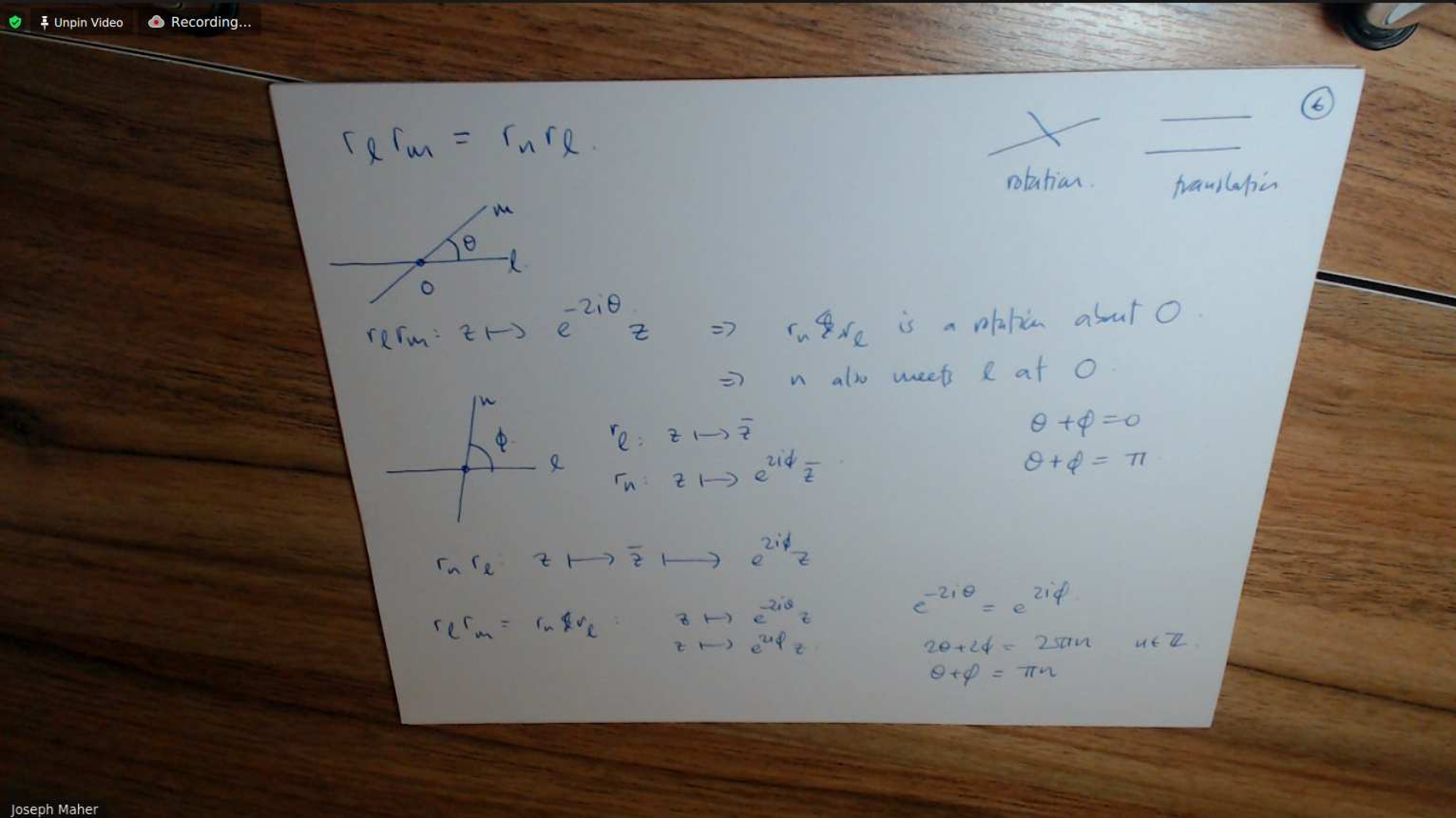
$r_m = \rho r_\ell \rho^{-1}$

wlog ℓ, m intersect at 0

$r_m \; z \longmapsto e^{-i\theta} z \longmapsto \overline{e^{-i\theta} z} \longmapsto e^{i\theta} e^{i\theta} \bar{z} = e^{2i\theta} \bar{z}$

$r_m : z \longmapsto e^{2i\theta} \bar{z}$

$r_\ell r_m : z \longmapsto e^{2i\theta} \bar{z} \longmapsto \overline{e^{2i\theta} \bar{z}} = e^{-2i\theta} z$



⑥

$r_\ell r_m = r_n r_\ell$.

$r_\ell r_m : z \mapsto e^{-2i\theta} z \Rightarrow r_n r_\ell$ is a rotation about O.

$\Rightarrow n$ also meets ℓ at O.

$r_\ell : z \mapsto \bar{z}$

$r_n : z \mapsto e^{2i\phi}\bar{z}$.

$\theta + \phi = 0$

$\theta + \phi = \pi$

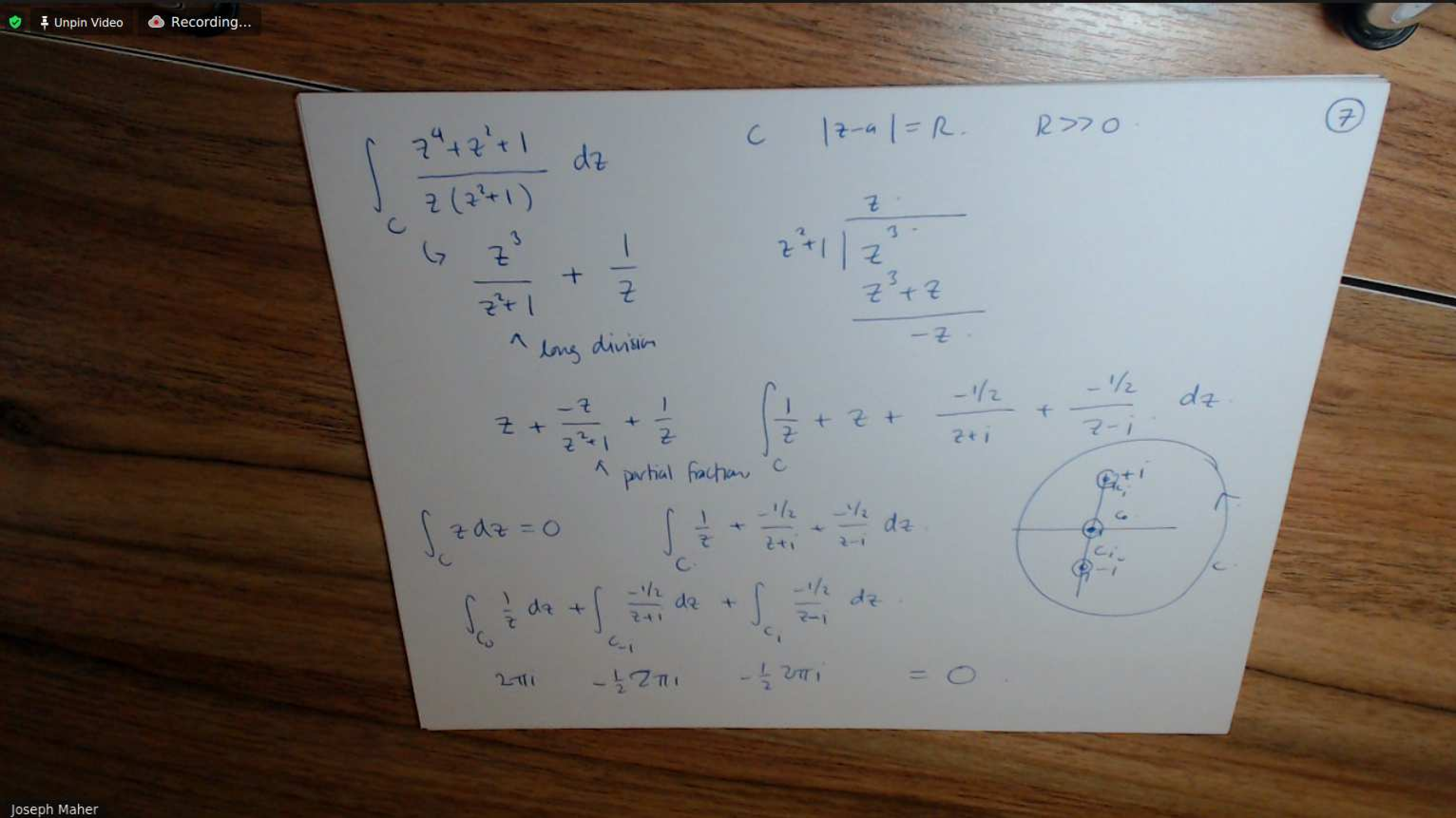
$r_n r_\ell \quad z \longmapsto \bar{z} \longmapsto e^{2i\phi} z$

$r_\ell r_m = r_n r_\ell : \quad z \mapsto e^{-2i\theta} z$
$\qquad z \mapsto e^{2i\phi} z$

$e^{-2i\theta} = e^{2i\phi}$.

$2\theta + 2\phi = 2\pi n \quad n \in \mathbb{Z}$.

$\theta + \phi = \pi n$



⑦

$$\int_C \frac{z^4+z^2+1}{z(z^2+1)}\,dz \qquad C \quad |z-a|=R. \qquad R>>0$$

$$\hookrightarrow \quad \frac{z^3}{z^2+1} + \frac{1}{z}$$

^ long division

$$\begin{array}{r|l} & z \\ \hline z^2+1 & z^3 \\ & z^3+z \\ \hline & -z \end{array}$$

$$z + \frac{-z}{z^2+1} + \frac{1}{z}$$

^ partial fraction

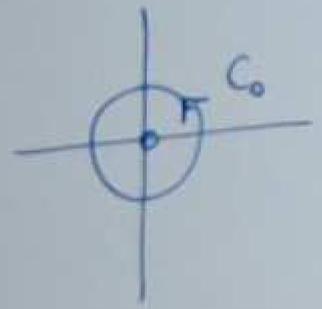
$$\int_C \frac{1}{z} + z + \frac{-1/2}{z+i} + \frac{-1/2}{z-i}\,dz$$

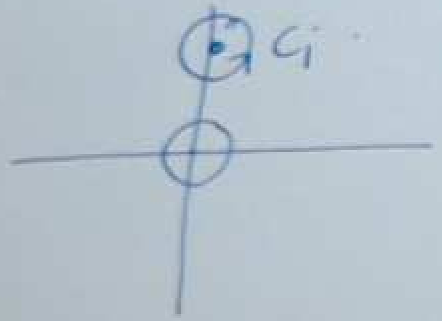
$$\int_C z\,dz = 0 \qquad \int_C \frac{1}{z} + \frac{-1/2}{z+i} + \frac{-1/2}{z-i}\,dz$$

$$\int_{C_0} \frac{1}{z}\,dz + \int_{C_{-1}} \frac{-1/2}{z+i}\,dz + \int_{C_1} \frac{-1/2}{z-i}\,dz$$

$$2\pi i \qquad -\tfrac{1}{2}2\pi i \qquad -\tfrac{1}{2}2\pi i \qquad = 0$$

$$\int_{c_0} \frac{1}{z}\,dz = 2\pi i$$





$$\int_{c_i} \frac{-1/2}{z-i}\,dz$$

explicitly change variable

$$w = z - i$$

$$\frac{dw}{dz} = 1$$

$$-\frac{1}{2}\int_{c_i} \frac{1}{z-i}\,dz$$

$$-\frac{1}{2}\int_{c_0} \frac{1}{w}\,dw$$

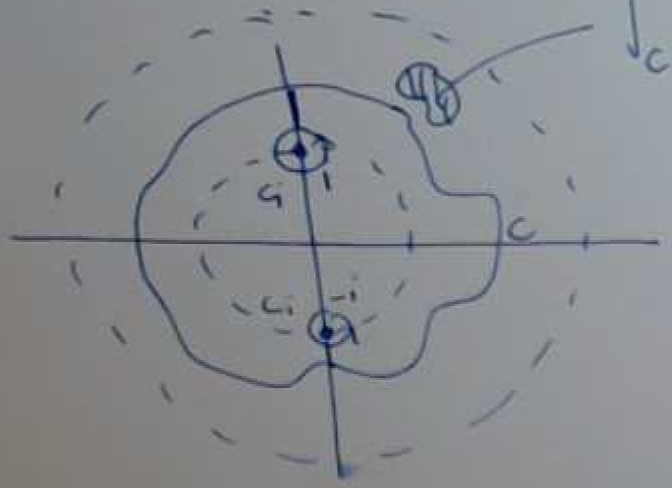
$$-\frac{1}{2}\, 2\pi i$$

$$\int_C \frac{1}{z^2+1}\,dz = \int_C \frac{+\frac{1}{2}i}{z+i} + \frac{-\frac{1}{2}i}{z-i}\,dz$$

$A+B=0$

$B-A=\frac{1}{i}=-i$

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} = \frac{A(z-i)+B(z+i)}{(z+i)(z-i)} = \frac{z(A+B)+i(B-A)}{z^2+1}$$



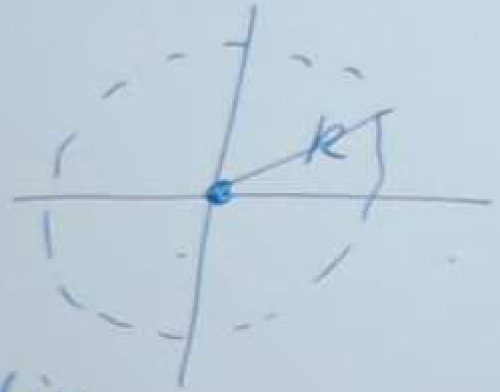
$\int_C \frac{1}{z^2+1}\,dz = 0$

$$\int_C \frac{1}{z^2+1}\,dz = \int_{C_{-i}} \frac{\frac{1}{2}i}{z+i}\,dz + \int_{C_i} \frac{-\frac{1}{2}i}{z-i}\,dz$$

$$\frac{1}{2}i\ 2\pi i + -\frac{1}{2}i\ 2\pi i = 0$$

last time: power series $\sum_{n=0}^{\infty} c_n z^n$

have radius of convergence



Remark if $f(z)$ is analytic in a disc, then it is given by a power series.

which power series? $f(0) + f'(0)z + f''(0)\frac{z^2}{2!} + f^{(3)}(0)\frac{z^3}{3!} + \cdots$

$$\left\{\begin{array}{c}\text{analytic}\\ \text{functions}\\ \text{in disc}\end{array}\right\} \underset{\uparrow}{\longleftrightarrow} \left\{\begin{array}{c}\text{power series}\\ \text{converging in } |z| < R\end{array}\right\}$$

bijection

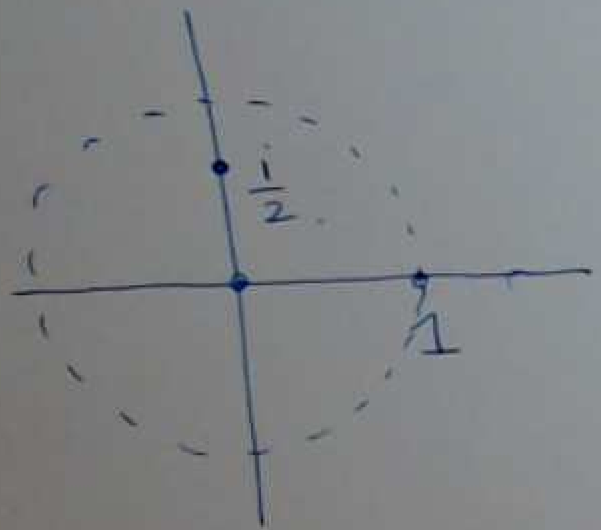
Example $$1+z+z^2+z^3+z^4+\cdots = \frac{1}{1-z}.$$

$|z|<1$

↗ analytic $|z|<1$.

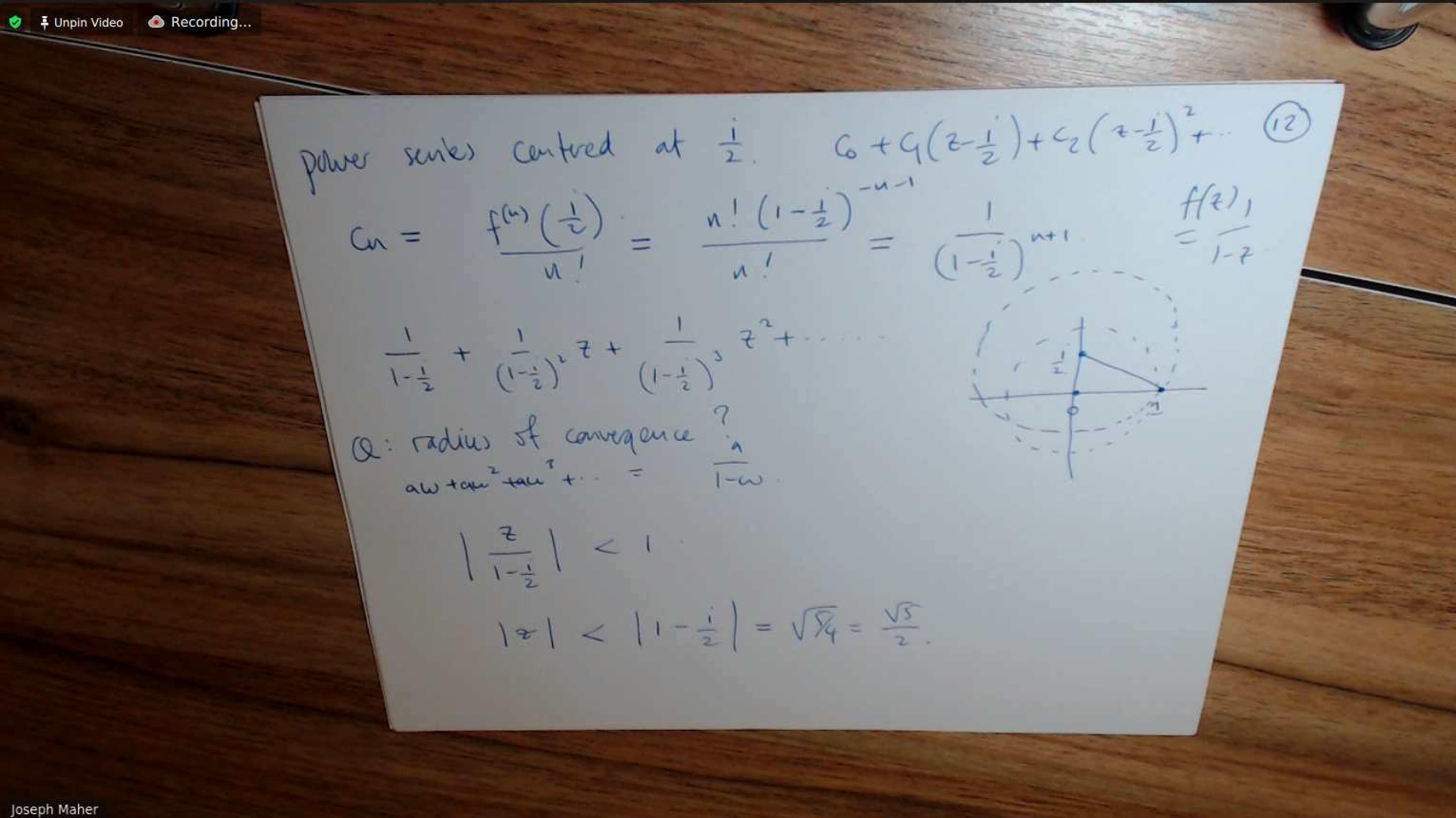
↖ analytic in $|z|<1$

analytic in $\mathbb{C}\setminus\{1\}$.



$$f(z) = (1-z)^{-1}$$
$$f'(z) = (1-z)^{-2}$$
$$f''(z) = 2(1-z)^{-3}$$
$$f^{(3)}(z) = 3!(1-z)^{-4}$$
$$\vdots$$
$$f^{(n)}(z) = n!(1-z)^{-n-1}$$
$$\vdots$$

$$c_n = \frac{n!}{n!}\cancel{z^n}$$
$$=1$$



power series centred at $\frac{i}{2}$. $c_0 + c_1(z-\frac{i}{2}) + c_2(z-\frac{i}{2})^2 + \cdots$ ⑫

$$c_n = \frac{f^{(n)}(\frac{i}{2})}{n!} = \frac{n!\,(1-\frac{i}{2})^{-n-1}}{n!} = \frac{1}{(1-\frac{i}{2})^{n+1}} \qquad f(z) = \frac{1}{1-z}$$

$$\frac{1}{1-\frac{i}{2}} + \frac{1}{(1-\frac{i}{2})^2} z + \frac{1}{(1-\frac{i}{2})^3} z^2 + \cdots\cdots$$

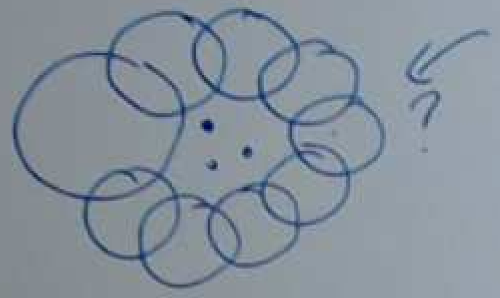
Q: radius of convergence?

$$aw + aw^2 + aw^3 + \cdots = \frac{a}{1-w}$$

$$\left|\frac{z}{1-\frac{i}{2}}\right| < 1$$

$$|z| < \left|1-\frac{i}{2}\right| = \sqrt{5/4} = \frac{\sqrt{5}}{2}.$$

Problem



Q: do they agree here?

A: not necessarily

§8 Special functions

recall $f : \mathbb{C} \longrightarrow \mathbb{C}$ is **entire** if it is analytic for all $z \in \mathbb{C}$. **e.g.** polynomials

Defn

$$\left.\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \end{aligned}\right\} \quad \begin{aligned} &R = +\infty \\ &\underline{\text{entire}} \end{aligned}$$

Prop$^{\underline{n}}$ $e^{z_1+z_2} = e^{z_1}e^{z_2} \quad \forall z_1 z_2 \in \mathbb{C}$.

Proof $e^{z_1} = 1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots \qquad e^{z_2} = 1 + z_2 + \frac{z_2^2}{2!} + \cdots$

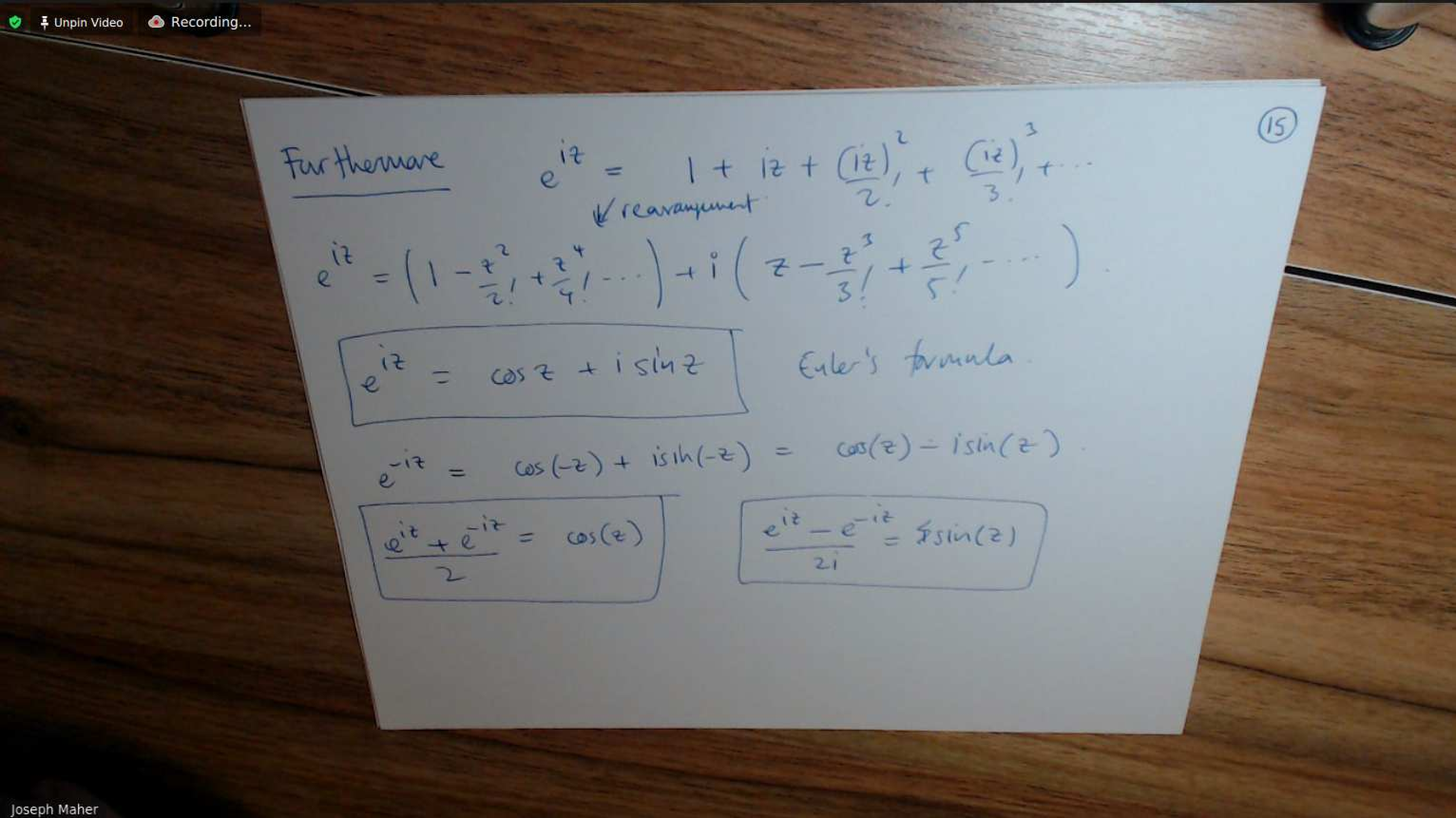
$$\left(1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots\right)\left(1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots\right). \qquad \binom{n}{k}.$$

$$1 + (z_1+z_2) + \left(\frac{z_1^2}{2!} + z_1z_2 + \frac{z_2^2}{2!}\right) + \left(\frac{z_1^3}{3!} + \overset{\checkmark}{\frac{z_1^2z_2}{2!}} + \frac{z_1z_2^2}{2!} + \frac{z_2}{3!}\right)$$

$$+ \frac{1}{2!}\left(z_1^2 + 2z_1z_2 + z_2^2\right) + \frac{1}{3!}\left(z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2\right)$$

$$+ \frac{1}{2!}(z_1+z_2)^2 + \frac{1}{3!}(z_1+z_2)^3 + \cdots$$

$$= e^{z_1+z_2} \qquad \square$$



Furthermore

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots$$

↙ rearrangement

$$e^{iz} = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right).$$

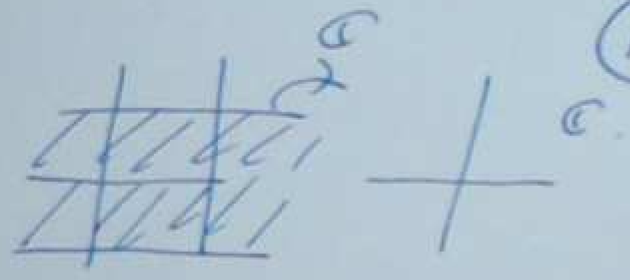
$$\boxed{e^{iz} = \cos z + i \sin z}$$ Euler's formula.

$$e^{-iz} = \cos(-z) + i\sin(-z) = \cos(z) - i\sin(z).$$

$$\boxed{\frac{e^{iz} + e^{-iz}}{2} = \cos(z)} \qquad \boxed{\frac{e^{iz} - e^{-iz}}{2i} = \cancel{i}\sin(z)}$$

useful facts

- e^z is periodic $e^{z+2\pi i} = e^z$.



$$\shortparallel$$

$$e^z \cdot \underbrace{e^{2\pi i}} \hookrightarrow \underset{1}{\cos(2\pi)} + i\underset{0}{\sin(2\pi)}$$

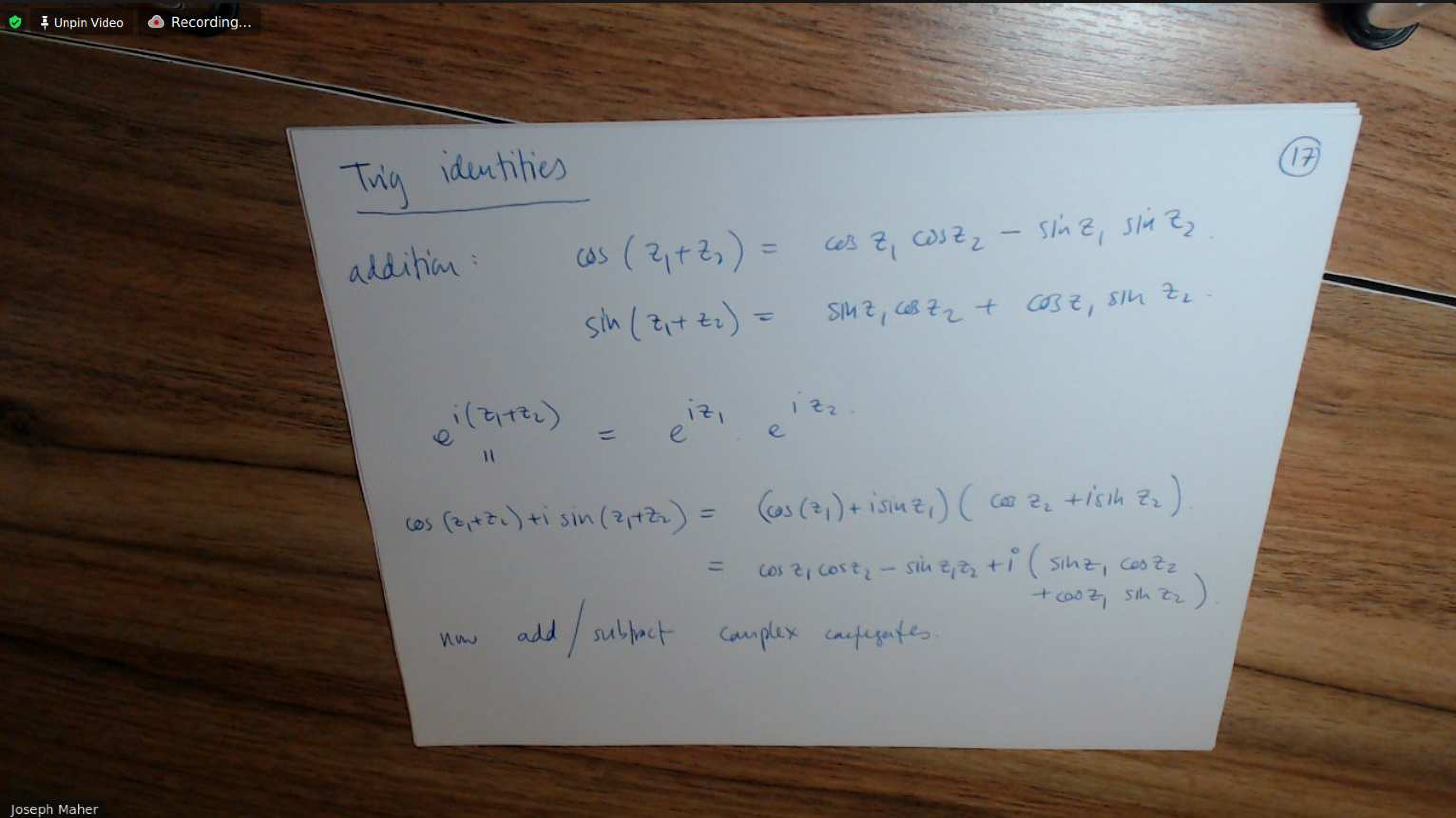
- polar form $z = x+iy$ $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$.

$$= re^{i\theta}.$$

$r = |e^z|$ $\theta = y$

- $e^z \neq 0$. $|e^z| = |e^x(\cos y + i\sin y)|$

$$= \underset{x\in\mathbb{R}}{|e^x|}. \qquad e^x \neq 0$$



Trig identities

addition: $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

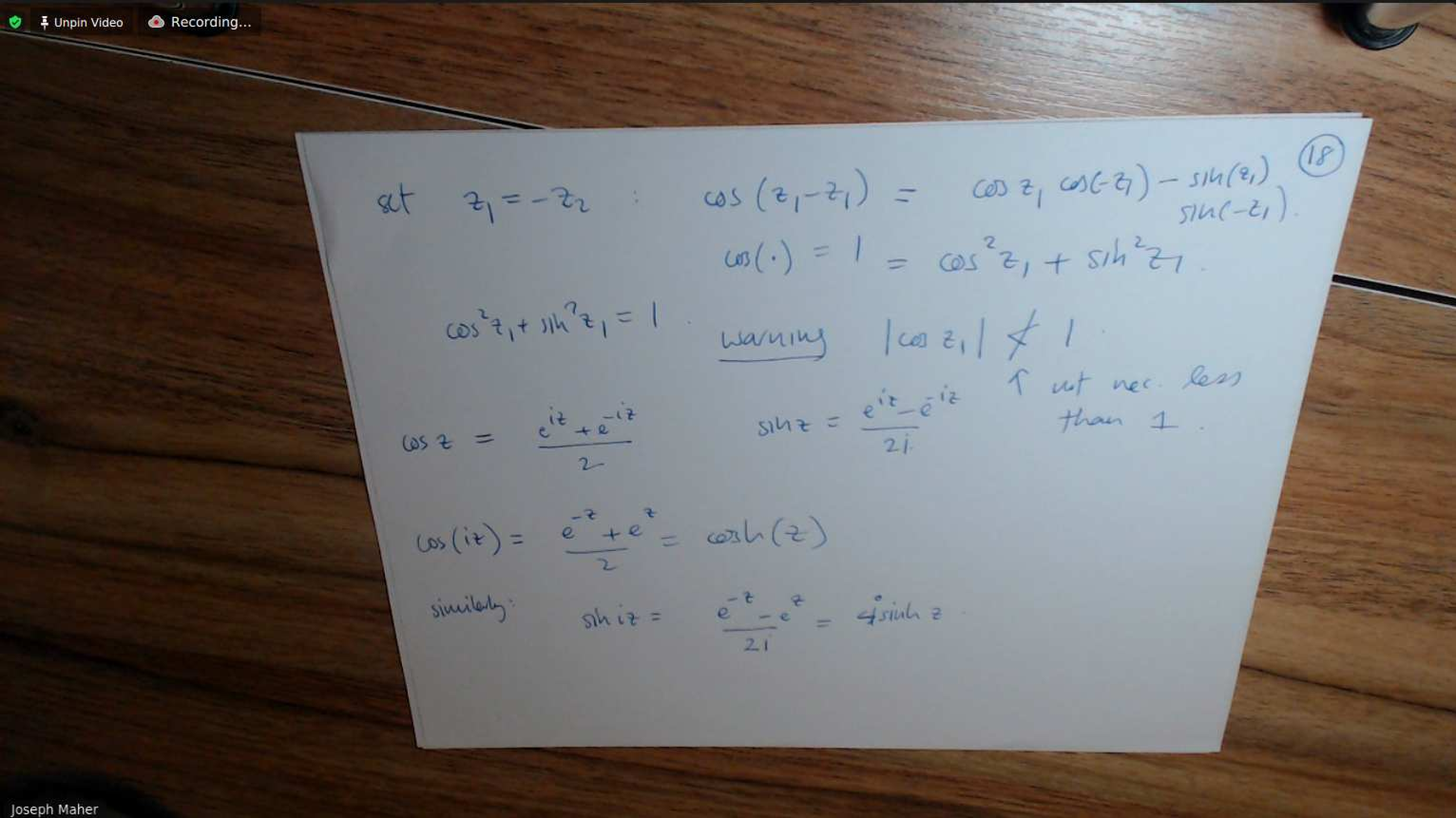
$$e^{i(z_1+z_2)} = e^{iz_1} \cdot e^{iz_2}.$$

$$\parallel$$

$$\cos(z_1+z_2) + i \sin(z_1+z_2) = (\cos(z_1) + i \sin z_1)(\cos z_2 + i \sin z_2)$$

$$= \cos z_1 \cos z_2 - \sin z_1 z_2 + i\left(\sin z_1 \cos z_2 + \cos z_1 \sin z_2\right)$$

now add / subtract complex conjugates.



⑱

set $z_1 = -z_2$: $\cos(z_1 - z_1) = \cos z_1 \cos(-z_1) - \sin(z_1)\sin(-z_1)$.

$\cos(\cdot) = 1 = \cos^2 z_1 + \sin^2 z_1$.

$\cos^2 z_1 + \sin^2 z_1 = 1$.

warning $|\cos z_1| \not< 1$

↑ not nec. less than 1.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(iz) = \frac{e^{-z} + e^{z}}{2} = \cosh(z)$$

similarly:

$$\sin iz = \frac{e^{-z} - e^{z}}{2i} = i \sinh z$$

$e^z \neq 0$.   Q: $\sin z = 0$ $\cos z = 0$?

⑲

solve $\sin z = 0$ $\frac{e^{iz} - e^{-iz}}{2i} = 0$

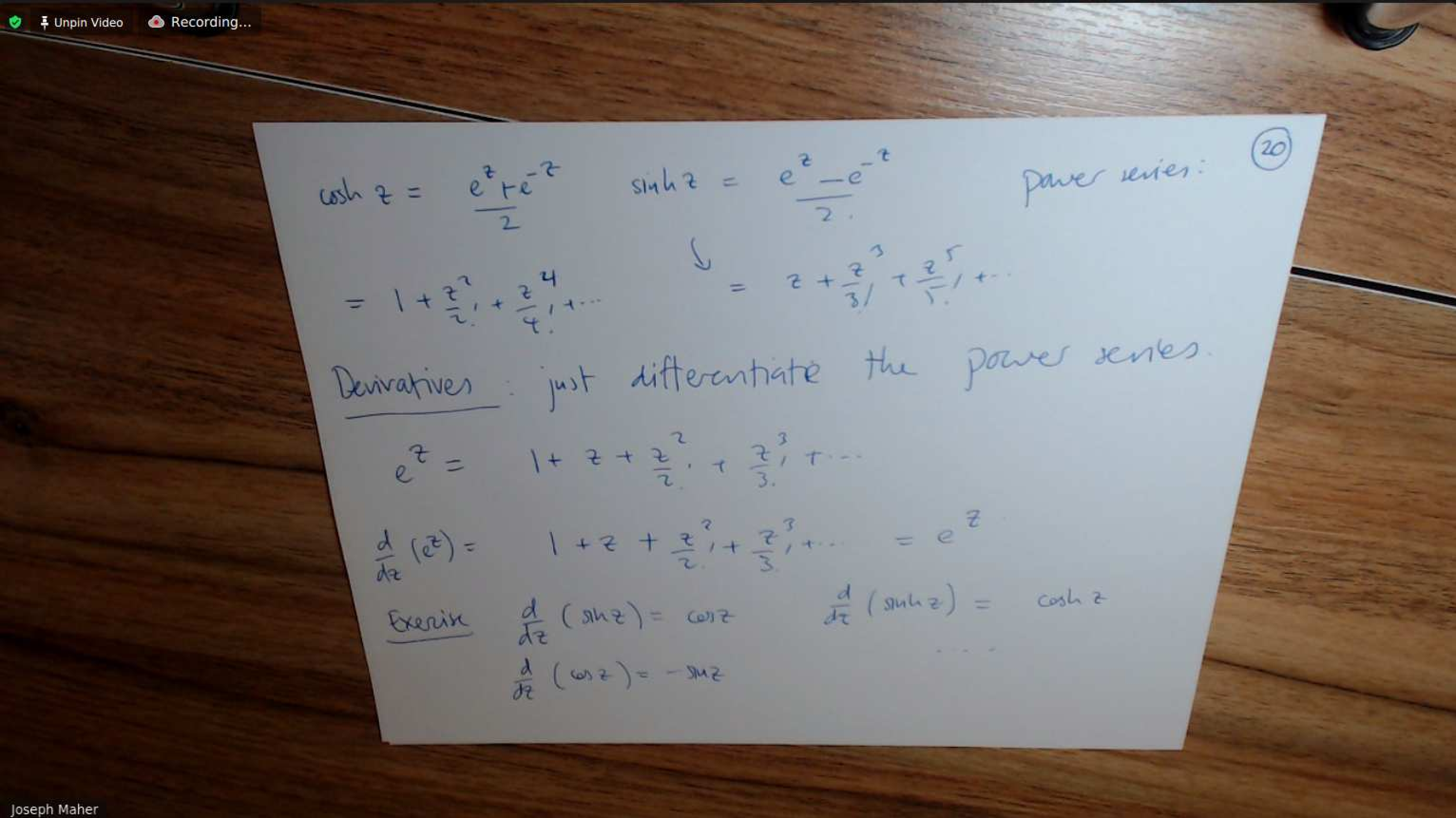
$$e^{iz} = e^{-iz}$$

$$e^{2iz} = \cancel{e}\, 1 = e^{2\pi n} \quad n \in \mathbb{Z}.$$

$\sin z = 0$

iff $z = \pi n$, $n \in \mathbb{Z}$

<u>Exercise</u> $\cos z = 0 \Leftrightarrow z = \frac{\pi}{2} + \pi n \quad n \in \mathbb{Z}$.



$\cosh z = \frac{e^z + e^{-z}}{2}$ $\sinh z = \frac{e^z - e^{-z}}{2}$. power series:

$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ $\downarrow$ $= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

Derivatives: just differentiate the power series.

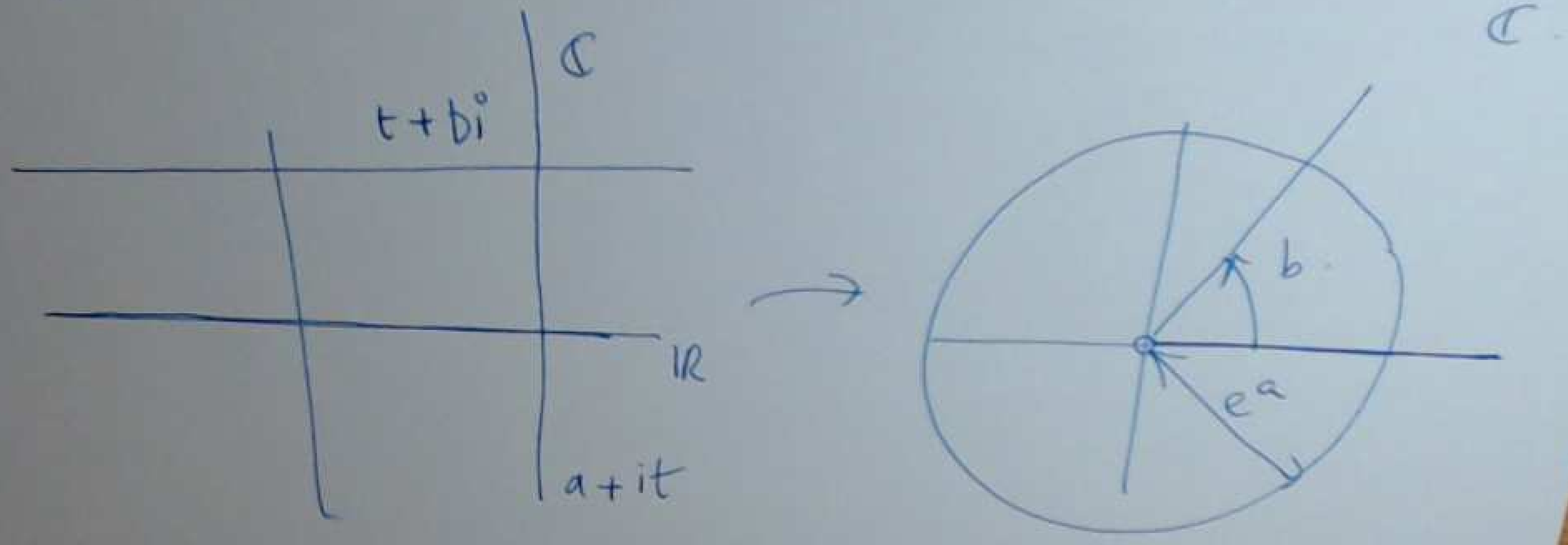
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\frac{d}{dz}(e^z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z$$

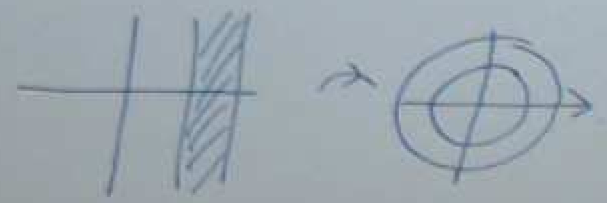
Exercise $\frac{d}{dz}(\sin z) = \cos z$ $\frac{d}{dz}(\sinh z) = \cosh z$

$\frac{d}{dz}(\cos z) = -\sin z$ ....

$f: z \longmapsto e^z$



$$t+bi \longmapsto e^{t+bi} = \underbrace{e^t}_{(0,\infty)} \underbrace{e^{bi}}_{\substack{\text{unit} \\ \text{complex number} \\ \text{of angle } b}}$$



$$a+it \longmapsto \underbrace{e^a}_{\substack{\text{fixed} \\ \text{number}}} \underbrace{e^{it}}_{\mathbb{C}} \ \substack{\text{runs} \\ \text{around unit circle}}$$