series $\sum z_n = z_1 + z_2 + z_3 + \cdots$

$\sum z_n$ is absolutely convergent if $\sum |z_n|$ convergent.

Absolutely convergent series can be rearranged and they still converge to the same thing.

<u>Lemma</u> Let $\sum z_n$ be absolutely convergent to z. and consider a family of series.

$$\left.\begin{array}{ll} Z_1 & z_{1,1} + z_{1,2} + \cdots \\ Z_2 & z_{2,1} + z_{2,2} + \cdots \\ Z_3 & z_{3,1} + \cdots \end{array}\right\}$$

each z_i appears exactly once in some Z_i.

then each Z_i is absolutely convergent, $z = \underbrace{Z_1 + Z_2 + \cdots}_{\text{absolutely convergent}}$

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(2)

Proof $z = \sum z_n$ is absolutely convergent, $\sum |z_n|$ converges.

$\Rightarrow$ each series $Z_1 = z_{n_{1,1}} + z_{n_{1,2}} + \cdots$ is absolutely convergent

furthermore $|Z_1 + Z_2 + \cdots + Z_n| \leq |z|$

$|z - (Z_1 + Z_2 + \cdots Z_n)|$ every z_i from $\sum_{n=1}^{\infty} z_n$ appears once in some Z_i

$\leq |z_m + z_{m+1} + \cdots| \longrightarrow 0$ as $m \to \infty$

sum of missing terms.

$\Rightarrow Z_1 + Z_2 + \cdots = z = \sum z_n$ $\square$

Joseph Maher

Th$\underline{\text{m}}$ (Rearrangement of series)

The terms of an absolutely convergent series can be rearranged without changing the sum.

Proof let $\sum_{n=1}^{\infty} z_n = s$

let $z_{n_1} + z_{n_2} + z_{n_3} + \cdots$ be a rearrangement

set $Z_k = z_{n_k}$ apply previous lemma $\square$.

Products of series

$z_1 + z_2 + z_3 + \cdots$

$z_1' + z_2' + z_3' + \cdots$

$$(z_1+z_2+\cdots)(z_1'+z_2'+\cdots) = z_1 z_1' + (z_1 z_2' + z_2 z_1') + (z_1 z_3' + z_2 z_2' + \cdots$$

Warning: for conditionally convergent series, the product series may not converge.

Thm (Multiplication of series) If $\sum z_i = s$, $\sum z_i' = s'$ are absolutely convergent series then their product is absolutely convergent, and sums to ss'. [say $\sum |z_n| = \sigma$ $\sum |z_n'| = \sigma'$]

Proof product series if $z_1 z_1' + z_1 z_2' + z_2 z_1' + z_2 z_2' + \cdots$

claim: this is absolutely convergent.

(*) $|z_1 z_1'| + |z_1 z_2'| + \cdots + |z_i z_j'|$ ← let n be the largest subscript appearing here, then (*)

$\leq (|z_1| + \cdots + |z_n|)(|z_1'| + \cdots + |z_n'|) \leq \sigma\sigma'$ ⇒ abs conv. □

<u>Warning</u>: for conditionally convergent series, the product series may not converge.

<u>Thm</u> (Multiplication of series) If $\sum z_i = s$, $\sum z_i' = s'$ are absolutely convergent series then their product is absolutely convergent, and sums to ss'. [say $\sum |z_n| = \sigma$ $\sum |z_n'| = \sigma'$]

<u>Proof</u> product series is $z_1 z_1' + z_1 z_2' + z_2 z_1' + z_2 z_2' + \cdots$

claim: this is absolutely convergent

$(*)$ $|z_1 z_1'| + |z_1 z_2'| + \cdots + |z_i z_j'|$ ← let n be the largest subscript appearing here, then $(*)$

$\leq (|z_1| + \cdots + |z_n|)(|z_1'| + \cdots + |z_n'|) \leq \sigma\sigma'$ $\Rightarrow$ abs conv $\square$

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abs. conv. $\Rightarrow$ can rearrange terms. (4)

$$z_1 z_1' + z_1 z_2' + z_1 z_3' + \cdots = z_1(z_1' + z_2' + \cdots) = z_1 s'$$

$$z_2 z_1' + z_2 z_2' + z_2 z_3' + \cdots = z_2(z_1' + z_2' + \cdots) = z_2 s'$$

$$z_3 z_1' + z_3 z_2' + \cdots = z_3(z_1' + z_2' + \cdots) = z_3 s'$$

$$= s'(z_1 + z_2 + z_3 + \cdots) = ss'. \quad \square$$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ ← can rearrange to converge to any real number.

fun fact $\sum z_n$ conditionally convergent series

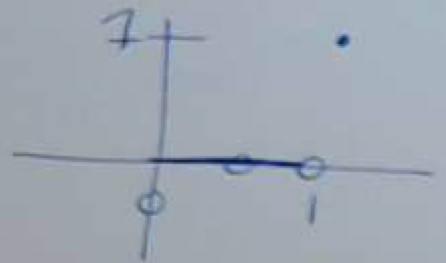
then either (1) can rearrange to converge to any $s \in \mathbb{C}$ (2) there is a line L s.t. you can rearrange to converge to any $s \in L$.

Joseph Maher

$f_n(x) = x^n$ defined $[0,1] \subseteq \mathbb{R}$.

define $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n$.

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$



(z^n) sequence.

write down a series for which this is the sequence of partial sums.

$z_1 = z$ $\qquad z_1 + z_2 = z^2$ $\qquad z_1 + z_2 + z_3 = z^3$.

$z_2 = z^2 - z$

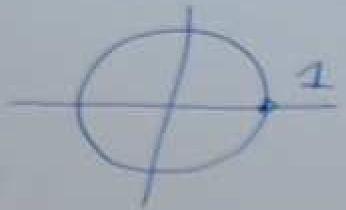
$z_3 = z^3 - z^2$ $\qquad z_k = z^k - z^{k-1}$

BAD

Example $\sum_{n=1}^{\infty} f_n(z)$ $\qquad f_n(z) = z^n - z^{n-1}$ 7

n-th partial sum $\quad z^n$

$$f(z) = \sum_{n=1}^{\infty} f_n(z) = \lim_{n\to\infty} (z^n) = \begin{cases} 0 & \text{if } |z|<1 \\ 1 & \text{if } z=1 \end{cases}$$



Q: when does a sum of functions converge to a continuous function?

A need uniform convergence

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Def^n: $\sum_{n=1}^{\infty} f_n(z)$ has partial sums $S_n(z) = f_1(z) + \cdots + f_n(z)$ (8)

assume $\sum_{n=1}^{\infty} f_n(z) = s(z)$ converges for all $z \in E \subseteq \mathbb{C}$.

then the sum is uniformly convergent in E

if for any $\epsilon > 0$ there is an $N(\epsilon) > 0$ s.t.

$|s(z) - s_n(z)| < \epsilon$ for all $n \geq N$ and for all $z \in E$.

key fact: N independent of $z \in E$ same N works for all points.

Joseph Maher

Example ① ~~then~~ $f_n(z) = z^n - z^{n-1}$

$s_n(z) = z^n$

$z^n \leftarrow$ not uniformly convergent on ~~[0,1]~~ $|z| < 1$

check $\left|\underset{0}{s(z)} - s_n(z)\right| = |z^n| < \epsilon$

$|z^n| = |z|^n < \epsilon$

$n \log|z| < \log(\epsilon)$

$n > \dfrac{\log(1/\epsilon)}{\log(1/|z|)} \to \infty$ as $|z| \to 1$

② $f_n(z) = z^n - z^{n-1}$

$s_n(z) = z^n$ consider this on $|z| \le r < 1$

this is uniformly convergent here.

$n > \dfrac{\log(1/\epsilon)}{\log(1/|z|)} \to \dfrac{\log(1/\epsilon)}{\log(1/r)}$

Thm If $s(z) = \sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in $E \subseteq \mathbb{C}$ and every $f_n(z)$ is continuous $z_0 \in E$ then $s(z)$ is continuous at z_0.

$\boxed{\text{Aside: proofs + refutations by Lakatos}}$

Proof 

let $z_0 + h \in E$.

$$s(z_0+h) - s(z_0)$$

$$= s(z_0+h) - s_N(z_0+h) + s_N(z_0+h) - s_N(z_0) + s_N(z_0) - s(z_0)$$

$$|s(z_0+h) - s(z_0)| \le |s(z_0+h) - s_N(z_0+h)| < \epsilon$$

uniform convergence
for all $\epsilon > 0$, there is a N s.t. $|s(z) - s_N(z)| < \epsilon$ for all $z \in E$

$$+ |s_N(z_0+h) - s_N(z_0)| < \epsilon$$

s_N c/s

$$+ |s_N(z_0) - s(z_0)| < \epsilon.$$

for all $\epsilon > 0$ there is a δ s.t. $|h| < \delta$, $|s_N(z_0+h) - s_N(z_0)| < \cancel{\delta}\,\epsilon$

for any $\epsilon > 0$, there is a $\delta > 0$ s.t. $|s(z_0+h) - s(z_0)| < \epsilon$ (11)

for all $|h| < \delta$. $\square$.

<u>Q</u>: how do you know if $\sum_{n=1}^{\infty} f_n(z) = s(z)$

is uniformly convergent?

<u>Thm</u> consider $s(z) = \sum_{n=1}^{\infty} f_n(z)$, suppose for all $n \in \mathbb{N}$,

$|f_n(z)| \leq a_n$ ~~for all $n \in \mathbb{N}$~~ such that $\sum a_n$

converges then $\sum f_n(z)$ is uniformly convergent.

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(12)

Proof spse $|f_n(z)| \leq a_n$ for all $z \in E$.

$s(z) = \sum f_n(z)$.

assume $\sum a_n$ converges.

Then $\sum |f_n(z)|$ converges (by comparison test) for all $z \in E$.

$\sum f_n(z)$ is absolutely convergent for all $z \in E$.

consider
$$\begin{aligned} |s(z) - s_n(z)| &\leq |f_{n+1}(z) + f_{n+2}(z) + \cdots| \\ &\leq |f_{n+1}(z)| + |f_{n+2}(z)| + \cdots \\ &\leq \underbrace{a_{n+1} + a_{n+2} + \cdots} \end{aligned}$$

for all $\epsilon > 0$, there is an N s.t. $\leq \epsilon$ for all $n \geq N$.

so we have $|s(z) - s_n(z)| \leq \epsilon$ for all $n \geq N$ for all $z \in E$ $\Rightarrow$ uniform convergence $\square$.

Joseph Maher

Example $f_n(z) = z^n - z^{n-1}$ $\qquad s_n(z) = z^n$.

$E: \quad |z| \leq r < 1$

to apply Th^m:

$$|f_n(z)| = |z^n - z^{n-1}| = |z^{n-1}|\,|z-1|.$$

$$\leq |z|^{n-1} 2 = r^{n-1} 2 = a_n.$$

$$\sum a_n = 2 + 2r + 2r^2 + \cdots = \frac{2}{1-r} < \infty$$

$$s(z) = \sum_{n=1}^{\infty} f_n(z).$$

Integration

recall: finite sums.

$$\int_C f_1(z) + \cdots + f_n(z)\, dz = \int_C f_1(z)\, dz + \cdots + \int_C f_n(z)\, dz.$$

infinite sums:

Thm Let $s(z) = \sum_{n=1}^{\infty} f_n(z)$ be a sum of continuous functions on C then

$$\int_C s(z)\, dz = \int_C f_1(z)\, dz + \int_C f_2(z)\, dz + \int_C f_3(z) + \cdots$$

as long as $\sum f_n(z)$ is uniformly convergent on C.

Proof $s(z) = \sum_{n=1}^{\infty} f_n(z)$ uniformly convergent on C (15)

cts $\Rightarrow$ integrable

consider $s_n(z) = f_1(z) + \dots + f_n(z)$ for any $\epsilon > 0$ there is an N s.t. $|s(z) - s_n(z)| < \epsilon$ for all $z \in C$.

$$\left| \int_C s(z) - s_n(z)\, dz \right| \leq \int_C |s(z) - s_n(z)|\, dz$$

$$\leq \epsilon \ell. \qquad \ell = \text{length of } C.$$

$$\Rightarrow \lim_{n\to\infty} \left| \int_C s(z) - s_n(z)\, dz \right| = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_C s(z) - s_n(z)\, dz = 0 \Rightarrow$$

$$\int_C s(z)\, dz = \lim_{n\to\infty} \int_C s_n(z)\, dz \qquad \square$$

Q: what about $\sum f_n(z)$ f_n analytic?

Thm (Weierstrass) $s(z) = \sum_{n=1}^{\infty} f_n(z)$

each $f_n(z)$ is analytic in ~~G~~ G, and the sum is uniformly convergent in ~~th~~ every closed bounded domain $\overline{D} \subseteq G$

open domain

Then $s(z)$ is analytic in G Furthermore, we can differentiate term by ~~term~~ term.

$$s^{(k)}(z) = f_1^{(k)}(z) + f_2^{(k)}(z) + f_3^{(k)}(z) + \cdots$$

this is uniformly convergent in every closed bounded domain ~~in~~ $\overline{D}$ in G.

$$(*) \int_{\gamma_R^{\not\subset}} \frac{k!}{2\pi i} \frac{s(z)}{(z-z_0)^{k+1}}\, dz = \int_{\gamma_R^{\not\subset}} \frac{k!}{2\pi i} \frac{f_1(z)}{(z-z_0)^{k+1}}\, dz + \int_{\gamma_R^{\not\subset}} \frac{k!}{2\pi i} \frac{f_2(z)}{(z-z_0)^{k+1}}\, dz + \cdots$$ 18

$\underline{k=0}$
$(0!=1)$

$$\int_{\gamma_R} \frac{s(z)}{z-z_0}\, dz = f_1(z_0) + f_2(z_0) + f_3(z_0) + \cdots$$
$$= s(z_0).$$

this shows Cauchy's integral holds for $s(z)$ inside γ_R

$\Rightarrow$ $s(z)$ analytic at z_0.

$\Rightarrow$ $s(z)$ infinitely differentiable

(*) for $k>0$

$$s^{(k)}(z) = \int_{\gamma_R} \frac{s(z)}{(z-z_0)^{k+1}} \frac{k!}{2\pi i}\, dz$$

$$= \int_{\gamma_R} \frac{k!}{2\pi i} \frac{f_1(z)}{(z-z_0)^{k+1}}\, dz + \int_{\gamma_R} \cdots \frac{f_2(z)}{\cdots}$$

remains to show uniform convergence of the differentiated series. $\square$

§7 Power series

A power series is an infinite sum of the form

$$c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots$$

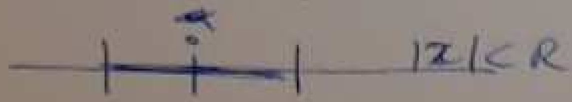
$c_i \in \mathbb{C}$ $\quad a \in \mathbb{C}$ called the center of the power series

from now on (at least for a bit) $a=0$.

$$c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

analogue of radius of convergence from real case is radius of convergence

$\mathbb{C}$



$|z|<R$

$|z|<R$