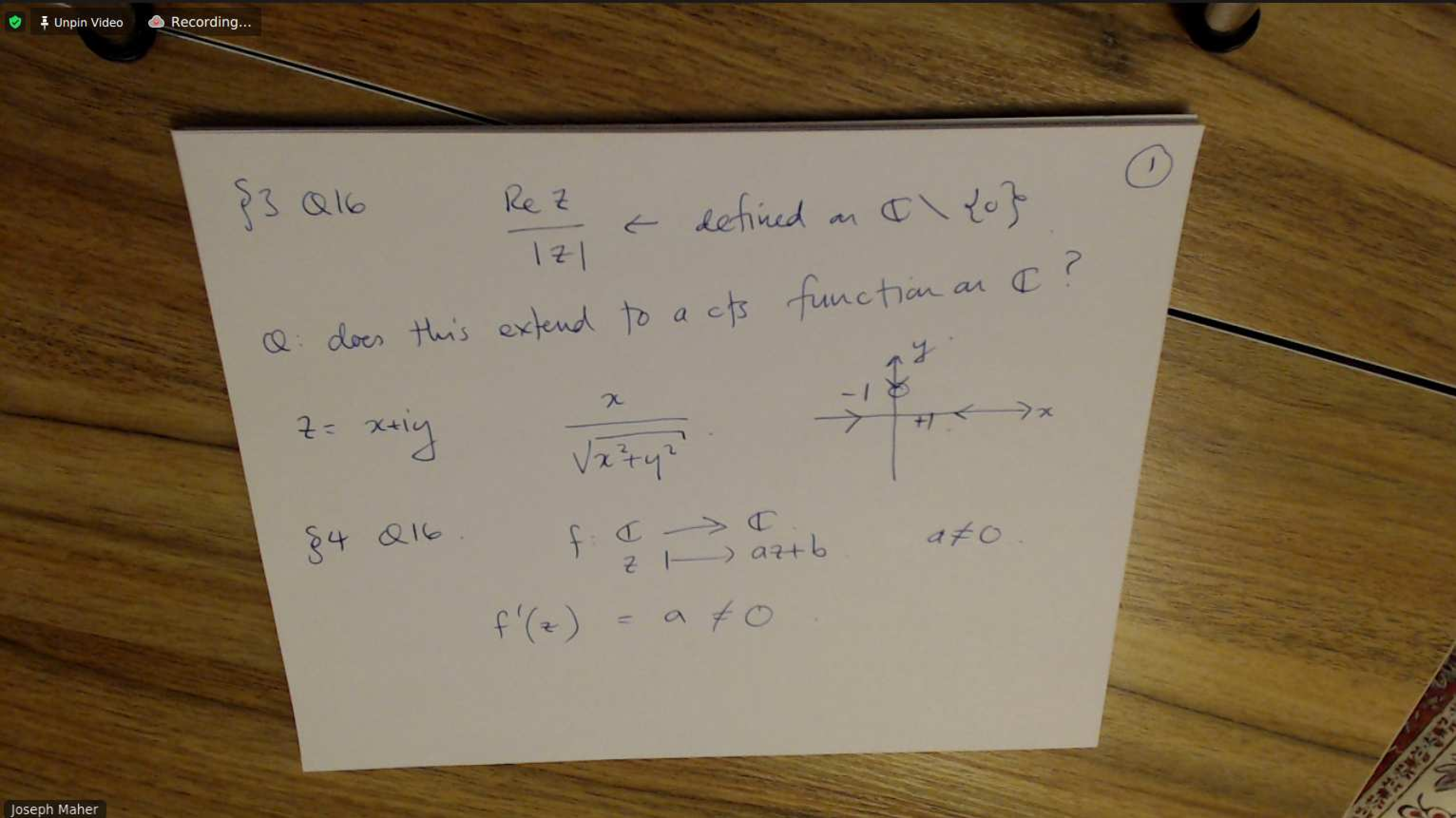

①

§3 Q16 $\frac{\operatorname{Re} z}{|z|}$ ← defined on $\mathbb{C} \setminus \{0\}$.

Q: does this extend to a cts function on $\mathbb{C}$?

$z = x+iy$ $\frac{x}{\sqrt{x^2+y^2}}$.

§4 Q16. $f: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto az+b$ $a \neq 0$.

$f'(z) = a \neq 0$.

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$f : \mathbb{C} \longrightarrow \mathbb{C}$.

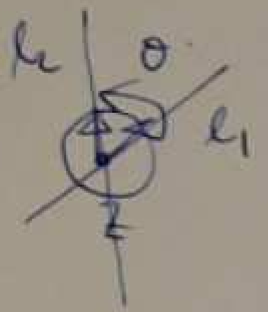
$z \longmapsto az+b$.

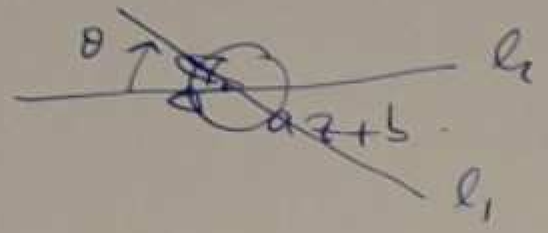
f^{-1}: $z \mapsto \frac{1}{a}(z-b)$

Th$^{\underline{m}}$: $f'(z) \neq 0$,

f analytic

f conformal.





$f'(z) \neq 0$

$z \mapsto \lambda z$

conformal.

f takes straight lines to straight lines.

l_1: $\quad Bz + \bar{B}\bar{z} + C = 0 \qquad B\frac{1}{a}(z-b) + \bar{B}\overline{\left(\frac{1}{a}(z-b)\right)} + C = 0$

l_2: $\quad Dz + \bar{D}\bar{z} + E = 0. \qquad D\frac{1}{a}(z-b) + \bar{D}\,\overline{\frac{1}{a}(z-b)} + E = 0$.

$B = \alpha + \beta i \qquad (\alpha+\beta i)(x+iy) + (\alpha-\beta i)(x-iy) \qquad \frac{1}{a}Bz + \frac{1}{\bar{a}}\bar{B}\bar{z}$.

$x(2\alpha) + y(-2\beta) \qquad -\alpha/\beta. \qquad \frac{1}{a}Dz + \frac{1}{\bar{a}}\bar{D}\bar{z}$

$D = \gamma + \delta i \qquad -\gamma/\delta$

same angle.

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§5.8 Harmonic functions

$u: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $u(x,y)$ is harmonic in a domain G if it has continuous second partial derivatives $\frac{\partial^2 u}{\partial x}, \underbrace{\frac{\partial^2 u}{\partial x y}, \frac{\partial^2 u}{\partial y \partial x}}_{\text{equal}}, \frac{\partial^2 u}{\partial y^2}$ at every point in G.

and satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

everywhere in G.

Let $u, v: \mathbb{R}^2 \to \mathbb{R}$ be two harmonic functions in G which satisfy the Cauchy-Riemann equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in G. Then u and v are conjugate harmonic functions.

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Thm Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined in G, then $f(z)$ is analytic iff ~~a~~ u and v are conjugate harmonic functions

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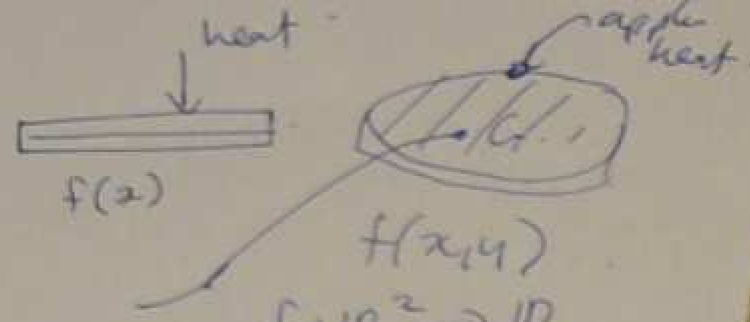
aside applications: heat flow

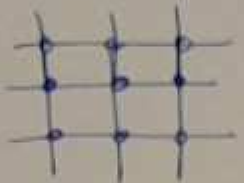


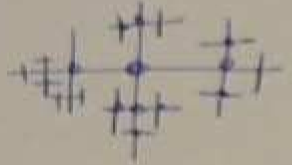
$f(x,y)$

$f: \mathbb{R}^2 \to \mathbb{R}$

inside f satisfies Laplaces equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

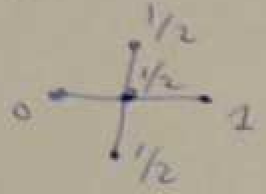
intuition: graph Laplacian

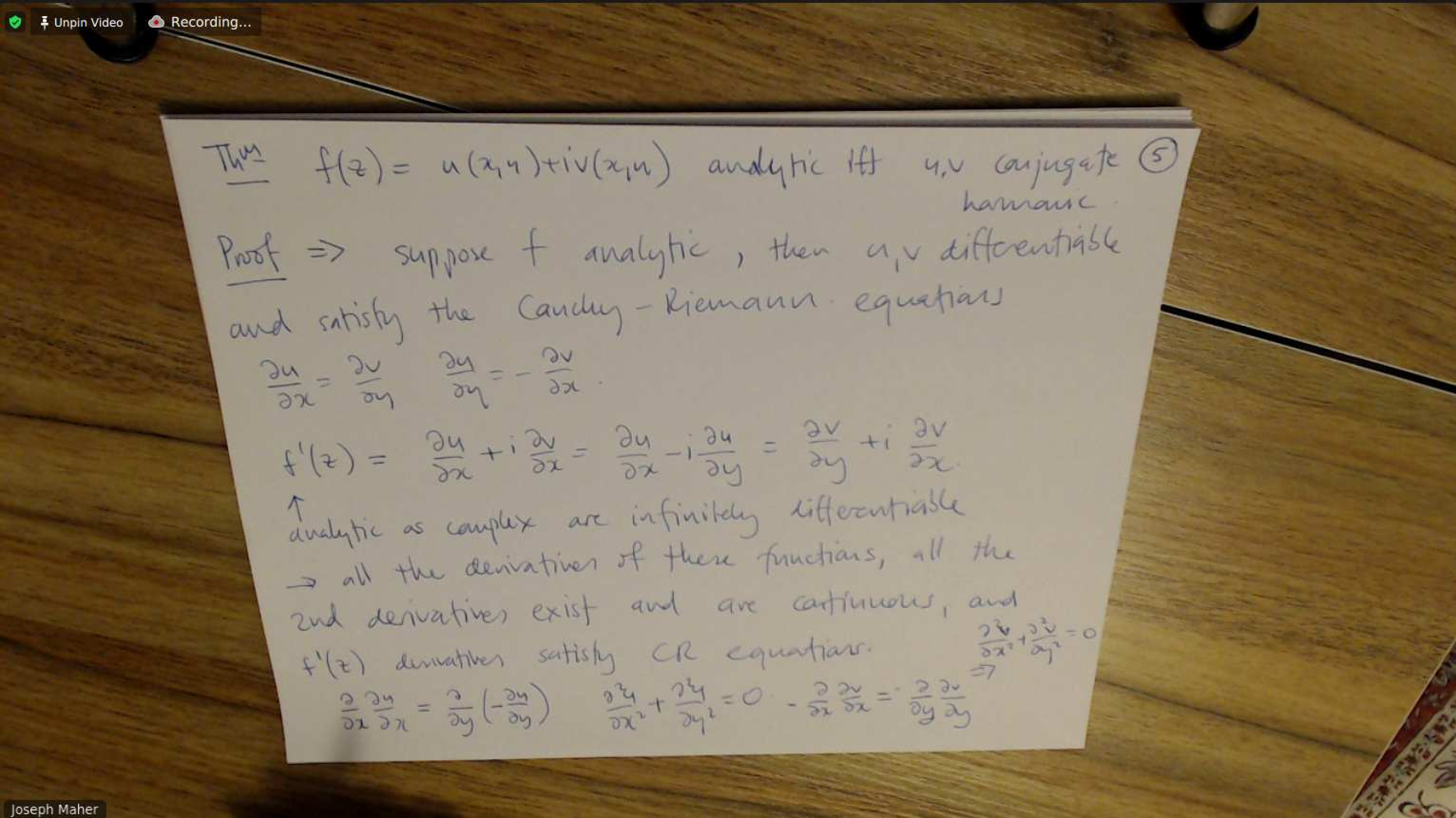




f: vertices $\to \mathbb{R}$

$f(x) =$ average over its neighbours in the graph





<u>Thm</u> $f(z) = u(x,y) + iv(x,y)$ analytic iff u, v conjugate harmonic. ⑤

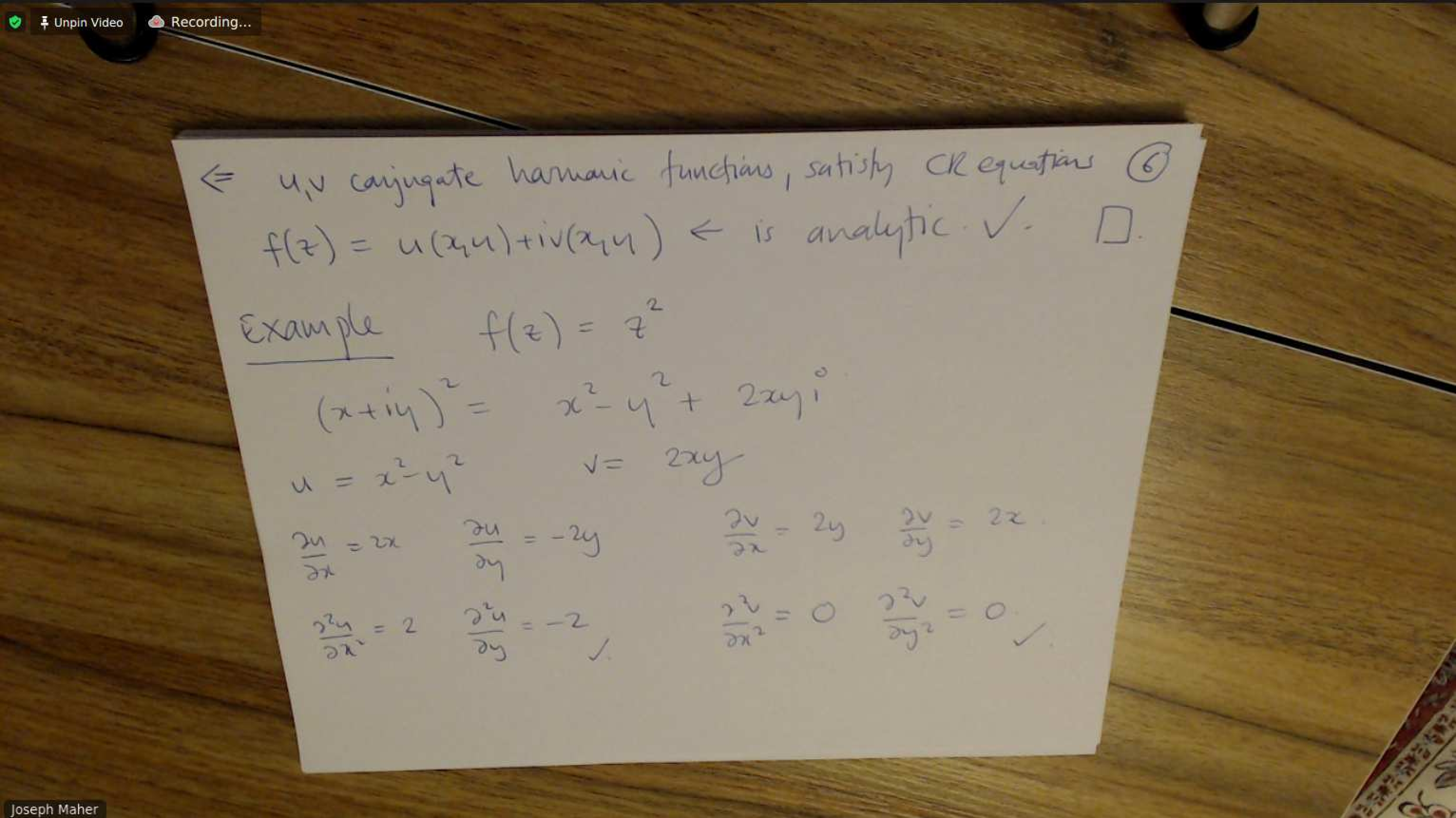
<u>Proof</u> $\Rightarrow$ suppose f analytic, then u, v differentiable and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

↑
analytic as complex are infinitely differentiable
→ all the derivatives of these functions, all the 2nd derivatives exist and are continuous, and $f'(z)$ derivatives satisfy CR equations.

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad -\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$



$\Leftarrow$ u, v conjugate harmonic functions, satisfy CR equations

$f(z) = u(x,y) + iv(x,y)$ $\leftarrow$ is analytic. ✓. $\square$.

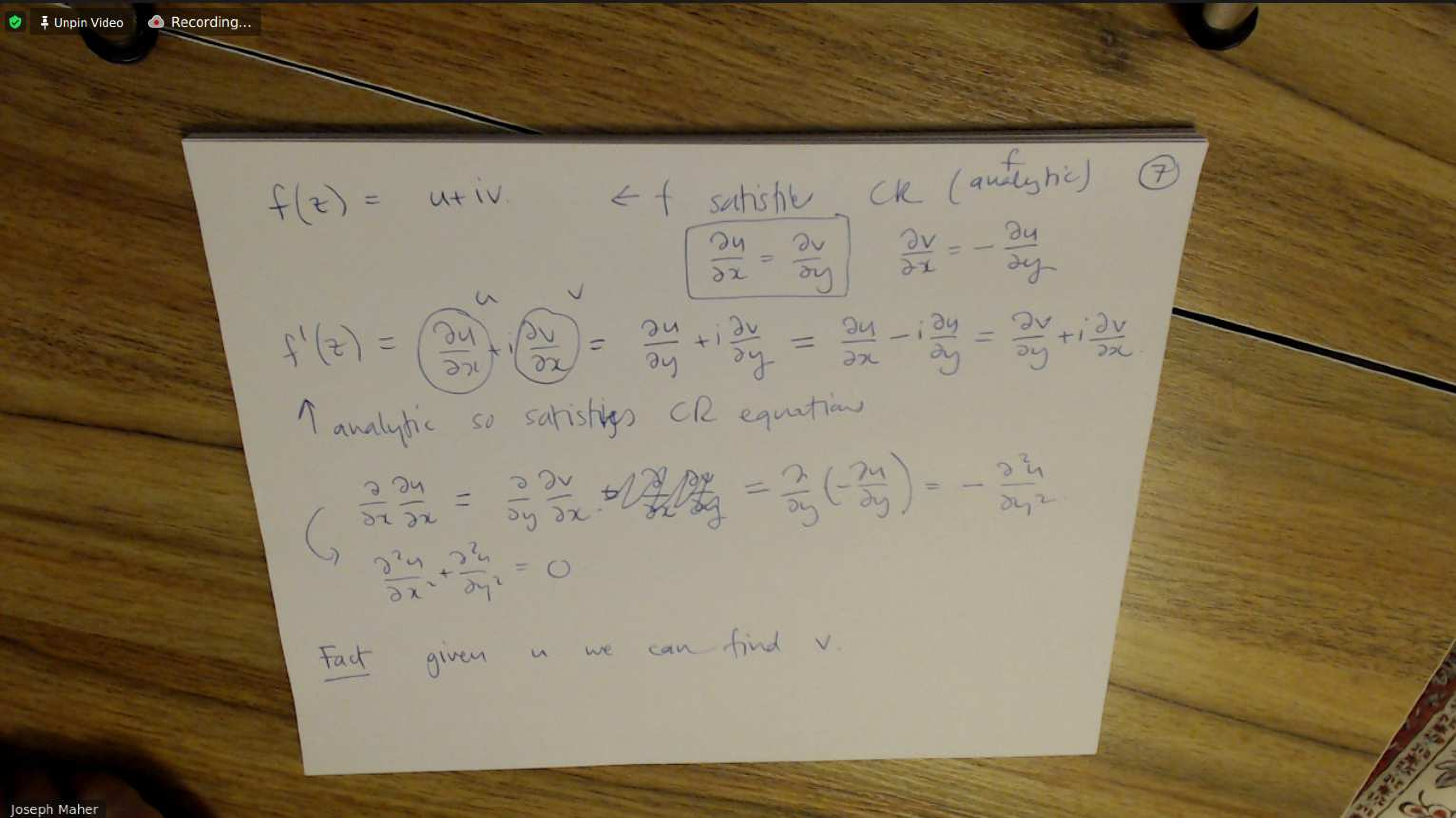
Example $f(z) = z^2$

$$(x+iy)^2 = x^2 - y^2 + 2xyi$$

$$u = x^2 - y^2 \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y} = -2 \ ✓ \qquad \frac{\partial^2 v}{\partial x^2} = 0 \qquad \frac{\partial^2 v}{\partial y^2} = 0 \ ✓$$



$f(z) = u + iv$ ← f satisfies CR ($\overset{f}{\text{analytic}}$)

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

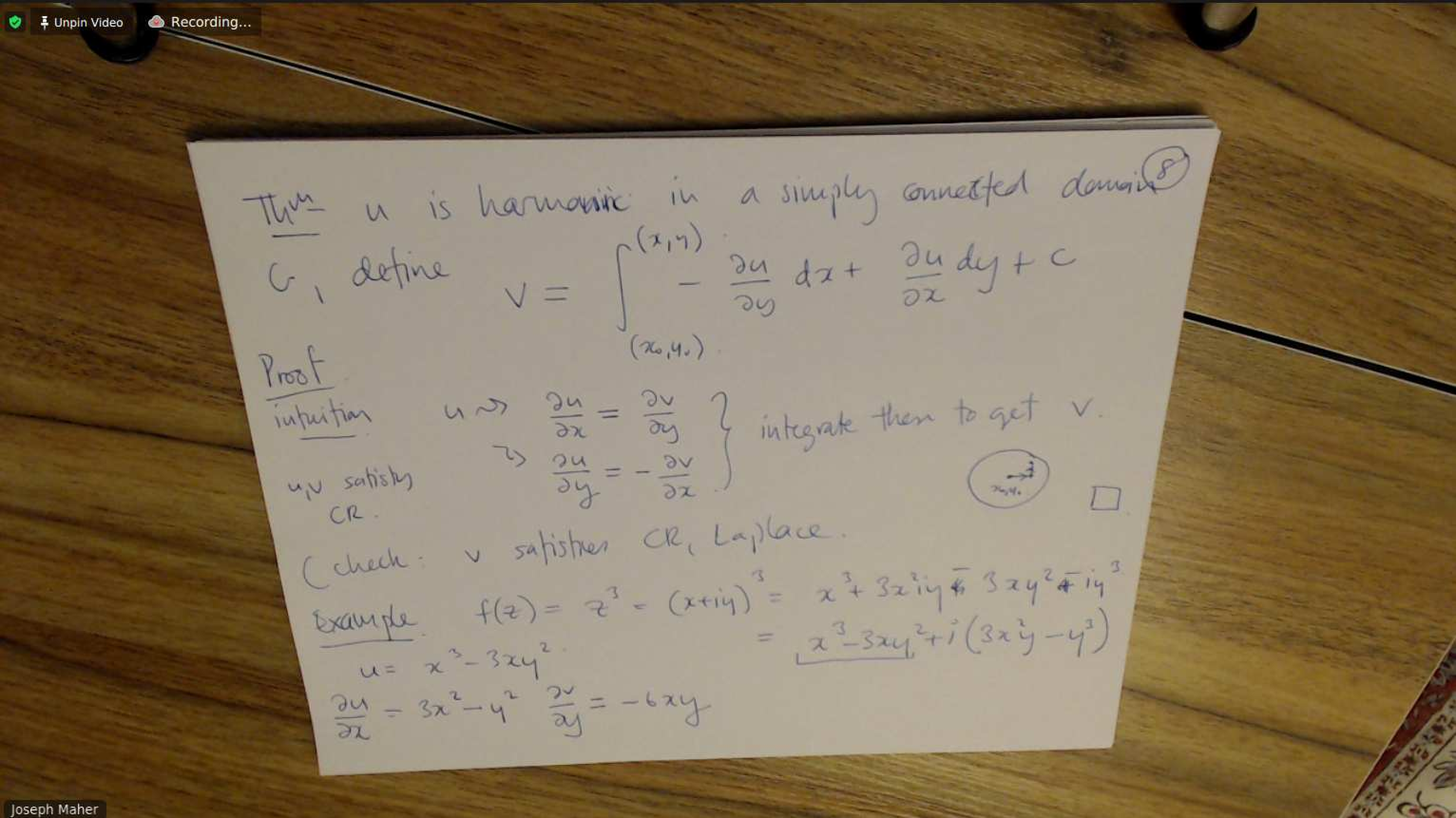
$$f'(z) = \overset{u}{\frac{\partial u}{\partial x}} + i\overset{v}{\frac{\partial v}{\partial x}} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial y}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

↑ analytic so satisfies CR equations

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \cancel{+\frac{\partial}{\partial x}\frac{\partial v}{\partial y}} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

<u>Fact</u> given u we can find v.



Th

Th$^{\underline{m}}$ u is harmonic in a simply connected domain G, define

$$V = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + c$$

Proof

intuition

$$u \rightsquigarrow \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \rightsquigarrow \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{integrate them to get } v.$$

u,v satisfy CR.

☐

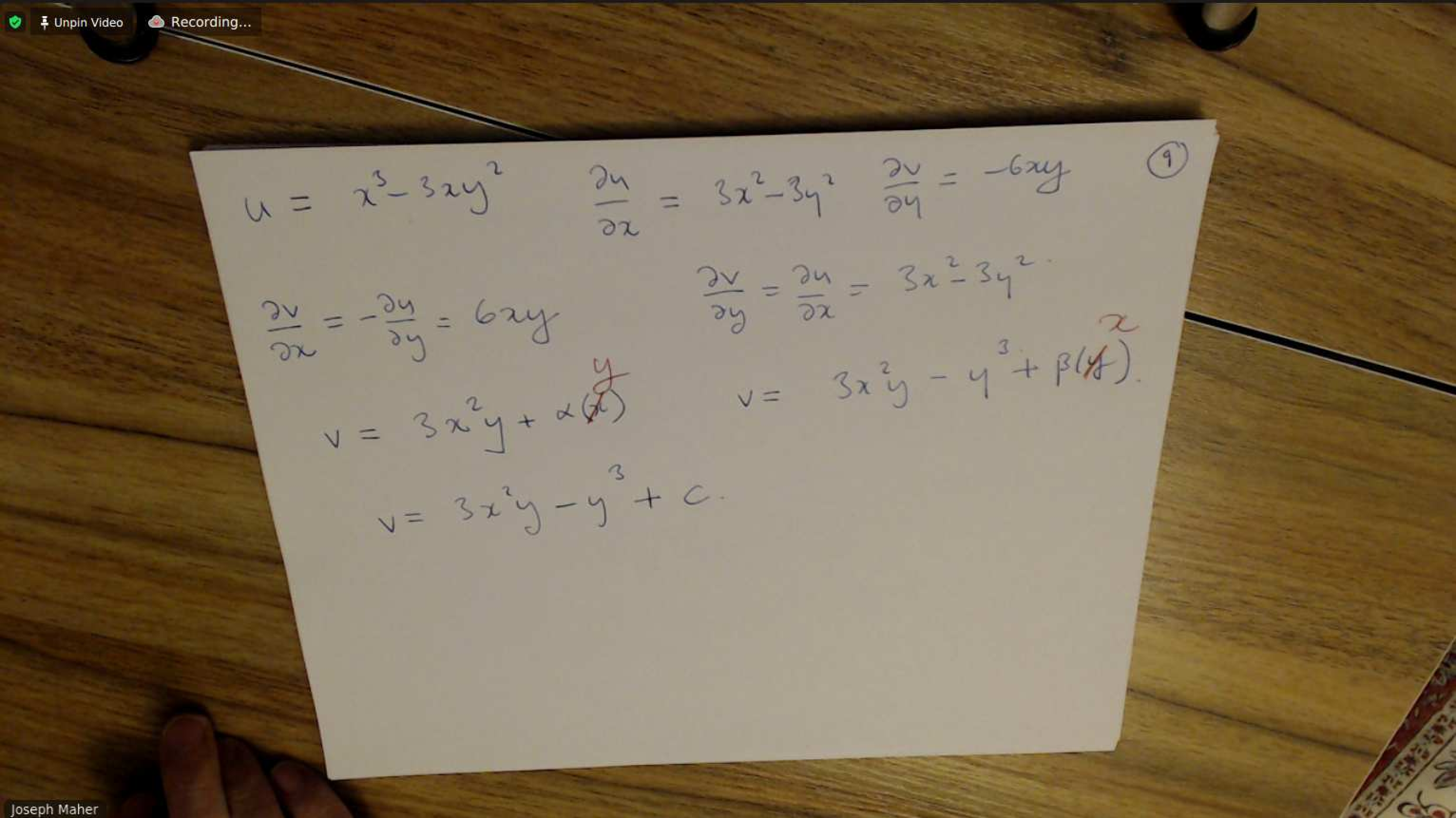
(check: v satisfies CR, Laplace.

Example $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2iy \overset{-}{\cancel{+}} 3xy^2 \overset{-}{\cancel{+}} iy^3$

$$= \underbrace{x^3 - 3xy^2}_{} + i(3x^2y - y^3)$$

$u = x^3 - 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 - y^2 \quad \frac{\partial v}{\partial y} = -6xy$



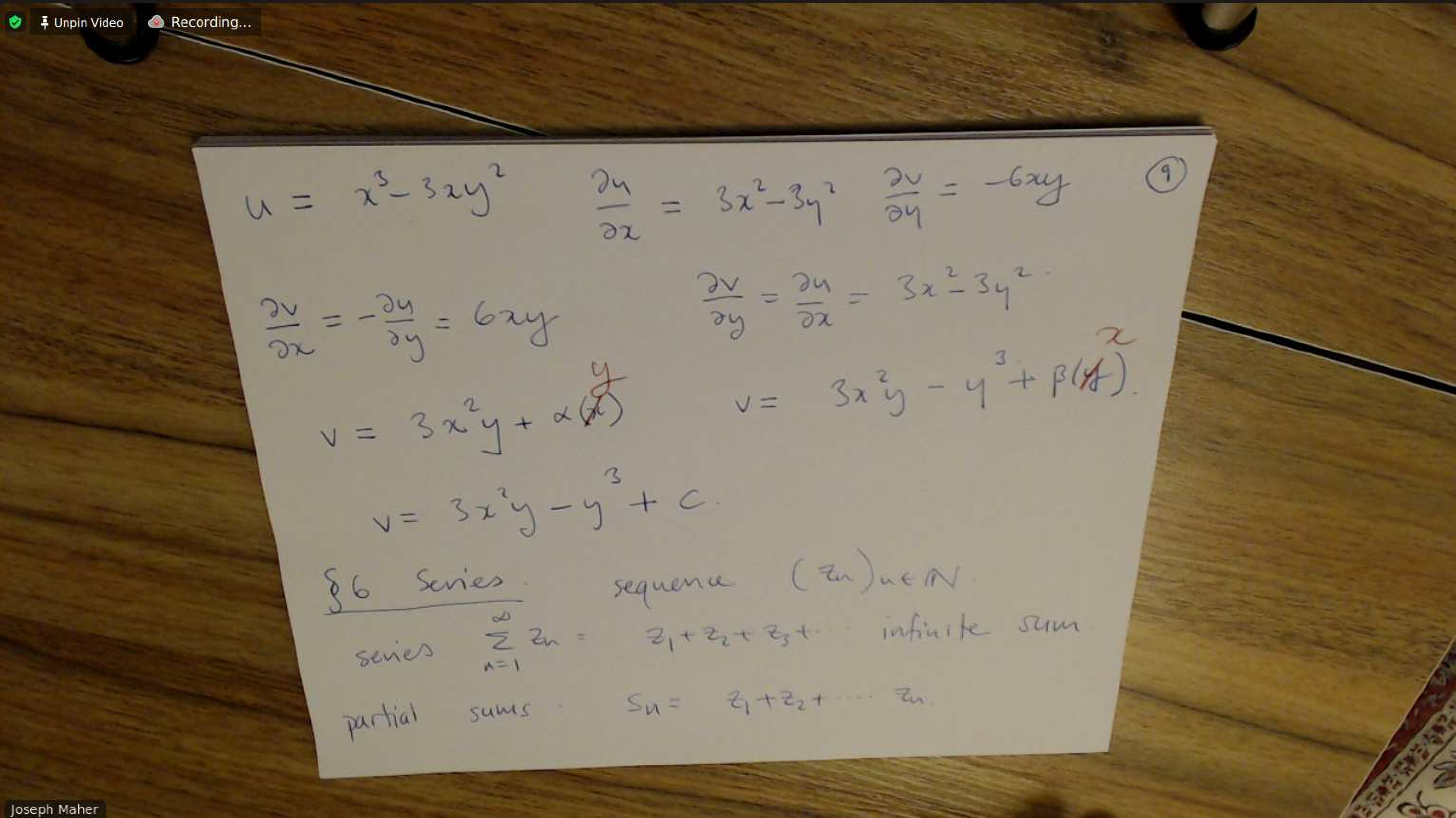
⑨

$u = x^3 - 3xy^2 \qquad \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial y} = -6xy$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2.$

$v = 3x^2y + \alpha(\overset{y}{\cancel{x}}) \qquad v = 3x^2y - y^3 + \beta(\overset{x}{\cancel{y}}).$

$v = 3x^2y - y^3 + c.$



$u = x^3 - 3xy^2$ $\quad \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\quad \dfrac{\partial v}{\partial y} = -6xy$

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = 6xy$ $\qquad \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$.

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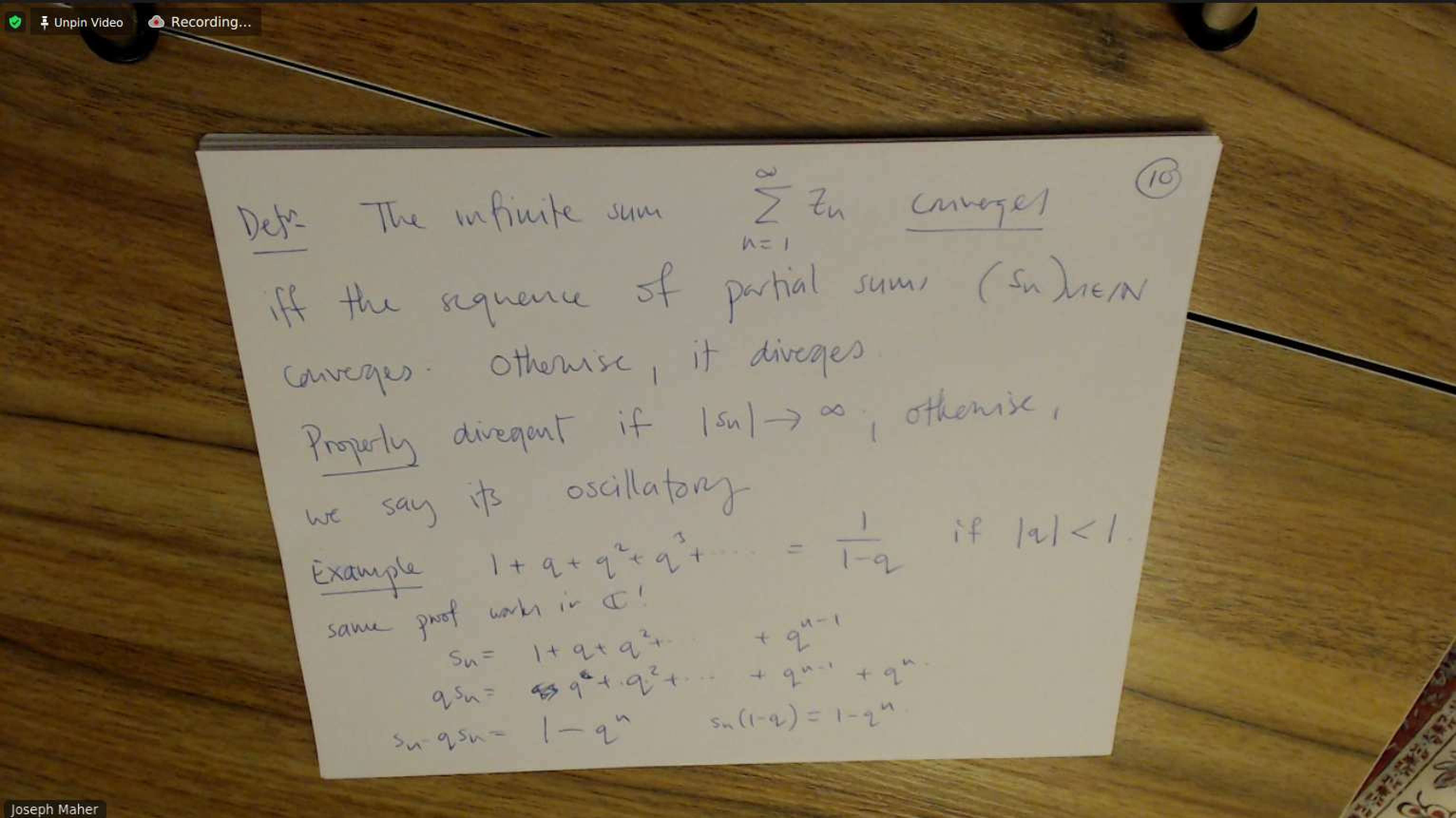
$v = 3x^2y - y^3 + c$.

§6 Series

sequence $(z_n)_{n \in \mathbb{N}}$.

series $\displaystyle\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots$ infinite sum

partial sums : $S_n = z_1 + z_2 + \cdots z_n$.



Def^n The infinite sum $\sum_{n=1}^{\infty} z_n$ **converges** iff the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ converges. Otherwise, it diverges.

Properly divergent if $|s_n| \to \infty$, otherwise, we say it's oscillatory

Example $1 + q + q^2 + q^3 + \cdots = \frac{1}{1-q}$ if $|q| < 1$.

same proof works in $\mathbb{C}$!

$$s_n = 1 + q + q^2 + \cdots + q^{n-1}$$
$$q s_n = q + q^2 + \cdots + q^{n-1} + q^n$$
$$s_n - q s_n = 1 - q^n \qquad s_n(1-q) = 1 - q^n$$

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$$S_n = \frac{1-q^n}{1-q}$$ as $n \to \infty$, if $|q| < 1$.

then $|q^n| \to 0$.

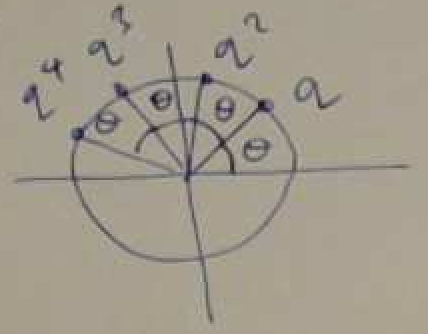
$\Rightarrow$ $q^n \to 0$

$$\lim_{n\to\infty} S_n = \frac{1}{1-q}.$$

remark doesn't converge for $|q| \geq 1$. $|q| > 1$.

$|q^n| \to \infty$.

Q what does q^n look like for $|q|=1$?



if $|q| = 1$

$|q^n| = |q|^n = 1$

θ rational ↙ multiple of π orbit periodic.

θ irrational multiple of π. orbit is dense in S^1 ← circle.

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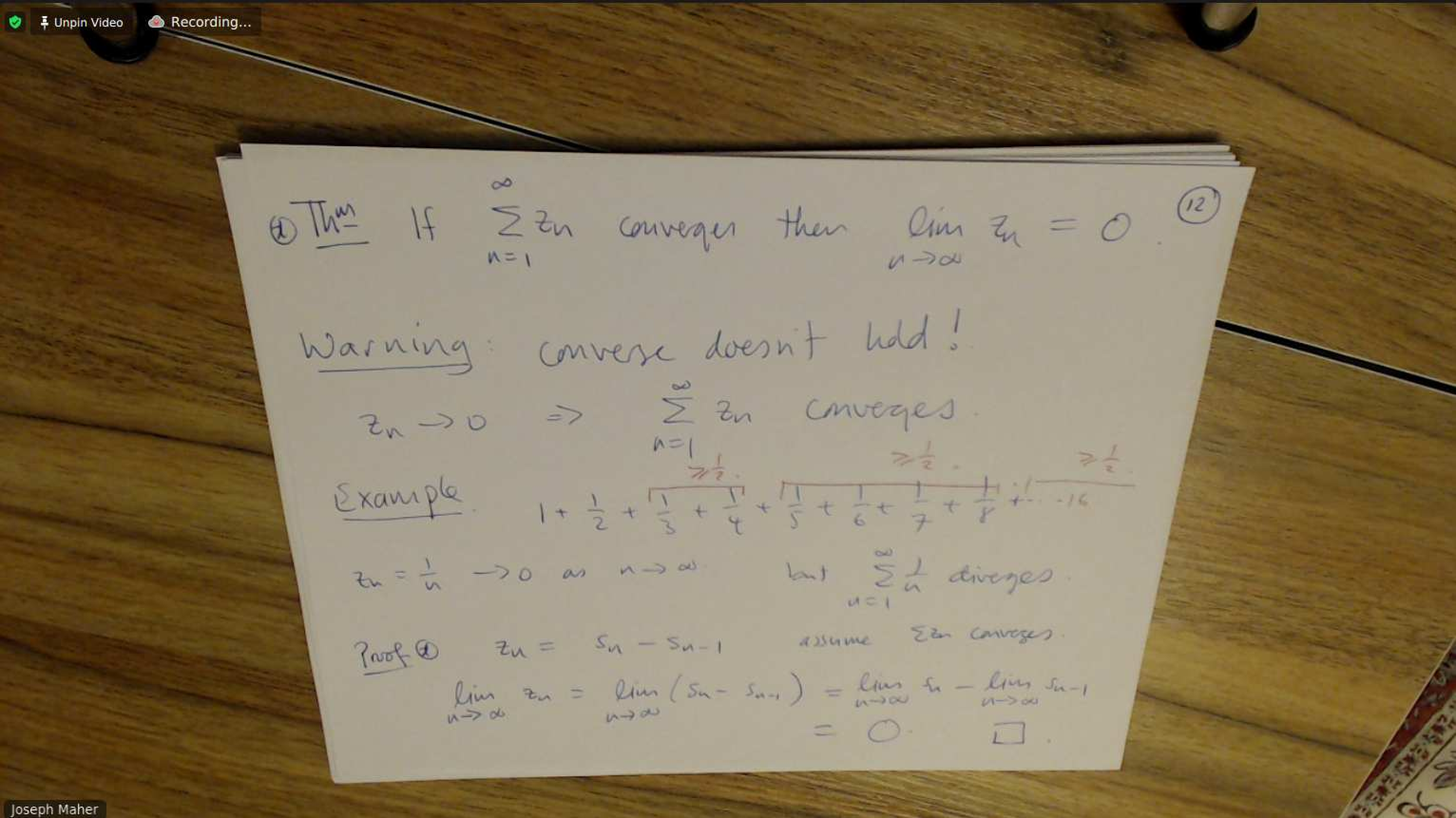


⑫

① **Thm** If $\sum_{n=1}^{\infty} z_n$ converges then $\lim_{n \to \infty} z_n = 0$.

Warning: converse doesn't hold!

$$z_n \to 0 \Rightarrow \sum_{n=1}^{\infty} z_n \text{ converges.}$$

Example

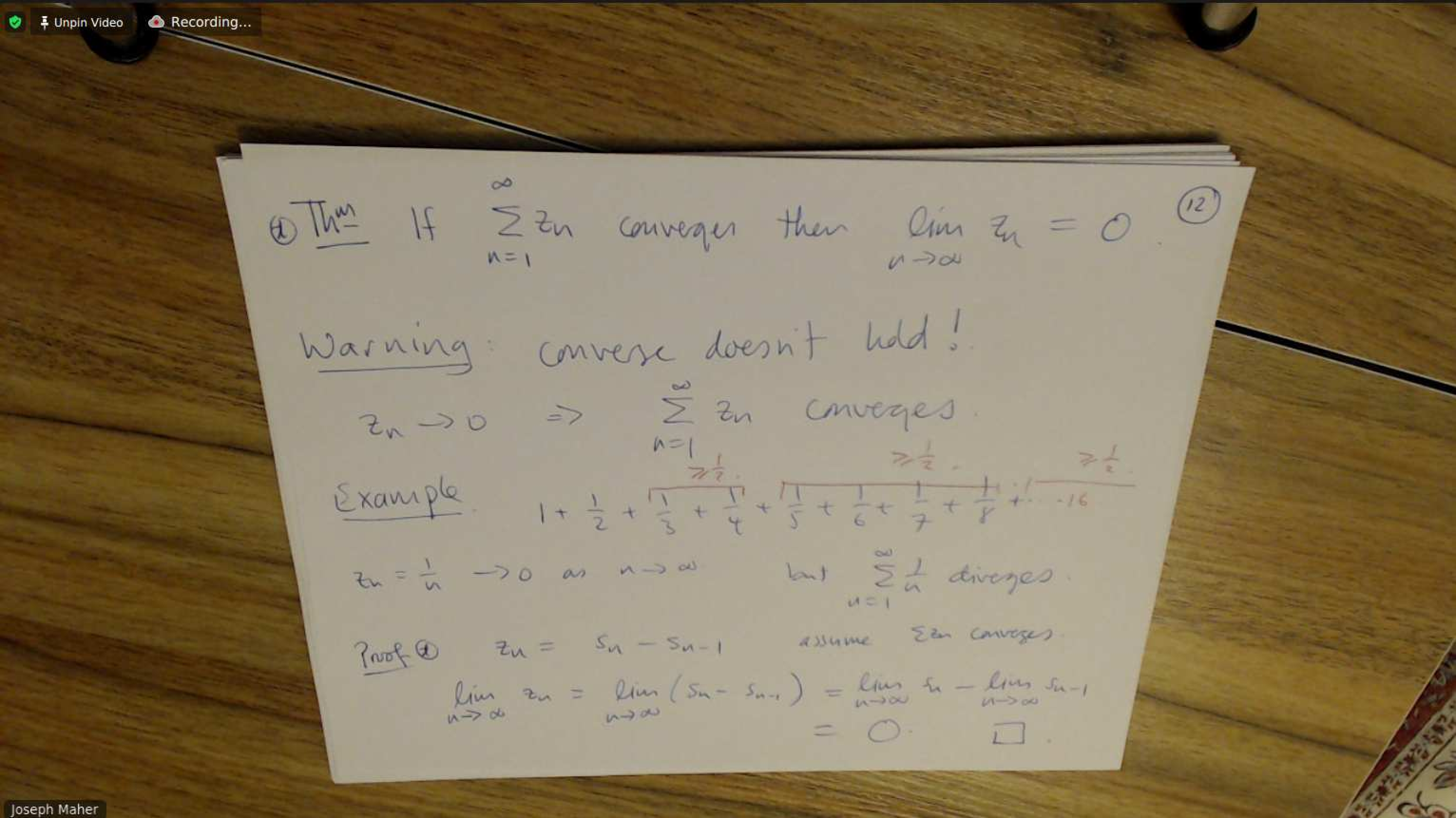
$$1 + \frac{1}{2} + \overbrace{\frac{1}{3} + \frac{1}{4}}^{\geq \frac{1}{2}} + \overbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}^{\geq \frac{1}{2}} + \overbrace{\frac{1}{9} + \cdots \frac{1}{16}}^{\geq \frac{1}{2}}$$

$z_n = \frac{1}{n} \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof ① $z_n = s_n - s_{n-1}$ assume $\sum z_n$ converges.

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} (s_n - s_{n-1}) = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1}$$

$$= 0. \quad \square$$



① **Thm** If $\sum_{n=1}^{\infty} z_n$ converges then $\lim_{n \to \infty} z_n = 0$.

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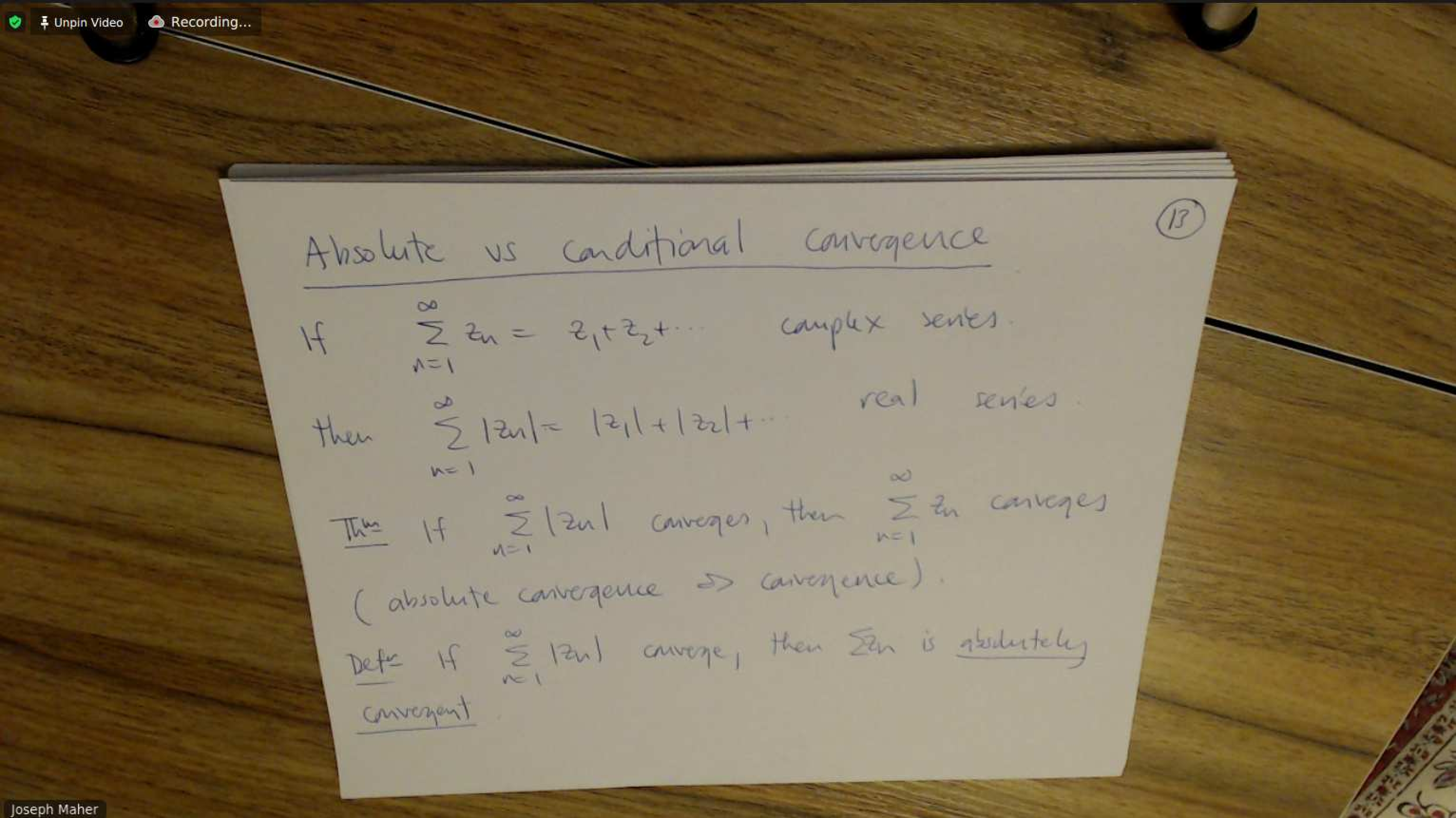
Example

$$1 + \frac{1}{2} + \overbrace{\frac{1}{3} + \frac{1}{4}}^{\geqslant \frac{1}{2}.} + \overbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}^{\geqslant \frac{1}{2}.} + \overbrace{\cdots \frac{1}{16}}^{\geqslant \frac{1}{2}.}$$

$z_n = \frac{1}{n} \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof ① $z_n = s_n - s_{n-1}$ assume $\sum z_n$ converges.

$$\lim_{n \to \infty} z_n = \lim_{n \to \infty} (s_n - s_{n-1}) = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1}$$
$$= 0. \qquad \square.$$



Absolute vs conditional convergence

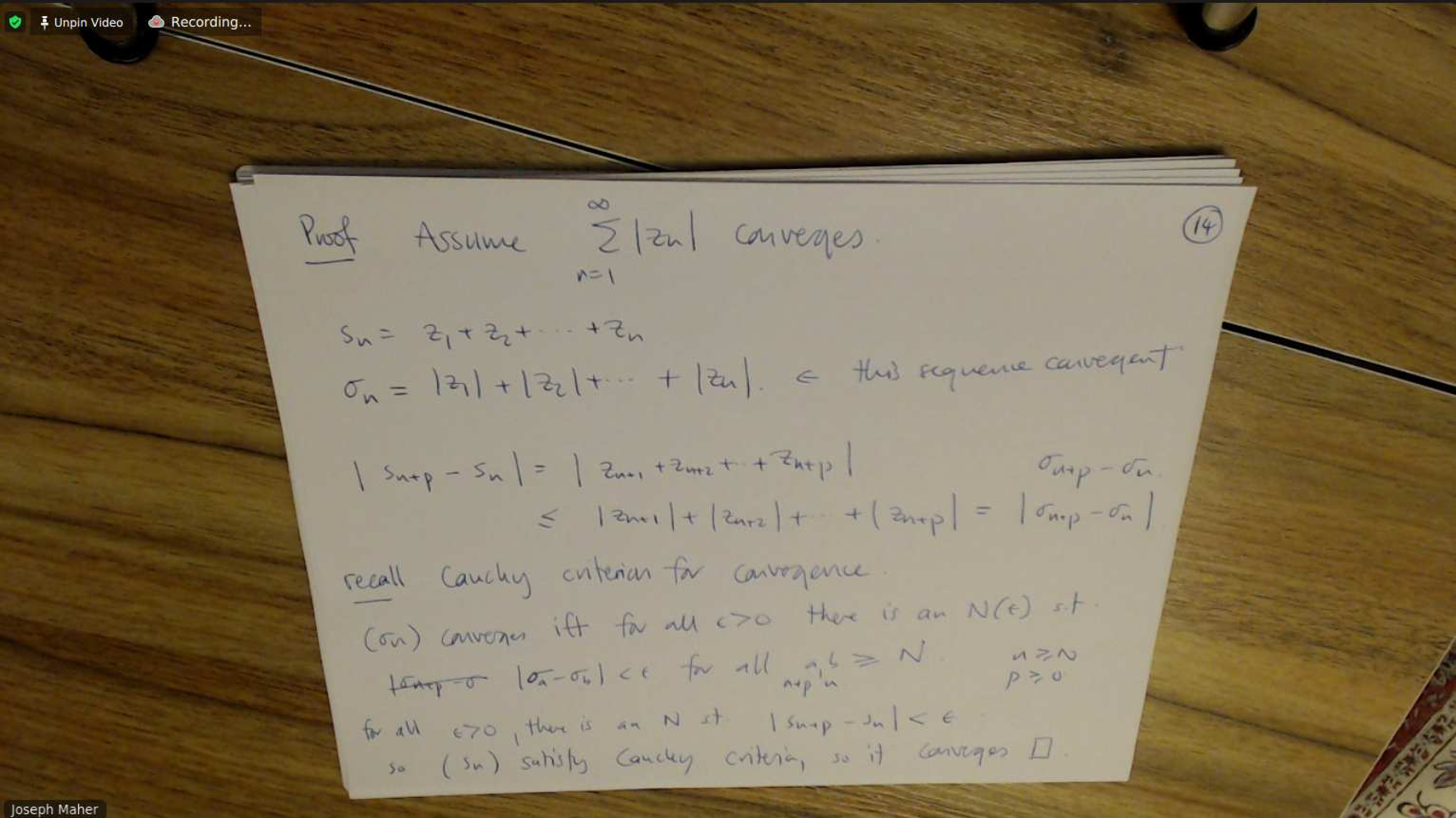
If $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$ complex series.

then $\sum_{n=1}^{\infty} |z_n| = |z_1| + |z_2| + \cdots$ real series.

Thm If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges

(absolute convergence $\Rightarrow$ convergence).

Defn If $\sum_{n=1}^{\infty} |z_n|$ converge, then $\sum z_n$ is *absolutely convergent*



(14)

Proof Assume $\sum_{n=1}^{\infty} |z_n|$ converges.

$s_n = z_1 + z_2 + \cdots + z_n$

$\sigma_n = |z_1| + |z_2| + \cdots + |z_n|$. ← this sequence convergent

$$|s_{n+p} - s_n| = |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| \qquad \sigma_{n+p} - \sigma_n$$
$$\le |z_{n+1}| + |z_{n+2}| + \cdots + |z_{n+p}| = |\sigma_{n+p} - \sigma_n|$$

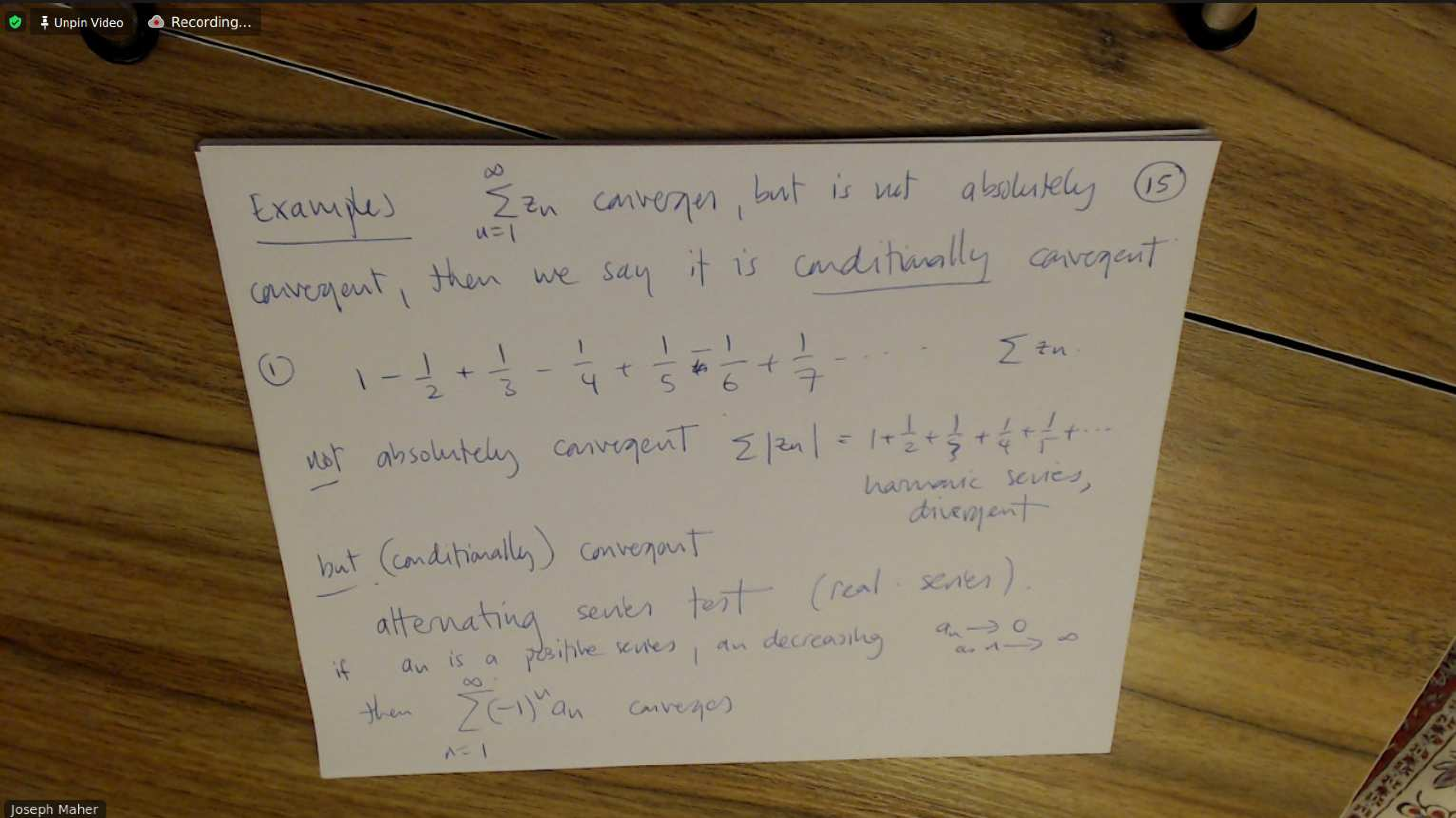
recall Cauchy criterion for convergence.

(σ_n) converges iff for all $\epsilon > 0$ there is an $N(\epsilon)$ s.t.

~~$|\sigma_{n+p} - \sigma$~~ $|\sigma_a - \sigma_b| < \epsilon$ for all $\underset{n+p}{a}, \underset{n}{b} \geq N$. $\quad n \geq N$, $p \geq 0$

for all $\epsilon > 0$, there is an N st $|s_{n+p} - s_n| < \epsilon$

so (s_n) satisfy Cauchy criteria, so it converges □



Examples $\sum_{n=1}^{\infty} z_n$ converges, but is not absolutely convergent, then we say it is conditionally convergent

① $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cancel{+} - \frac{1}{6} + \frac{1}{7} - \cdots$ $\sum z_n$

not absolutely convergent $\sum |z_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{1} + \cdots$

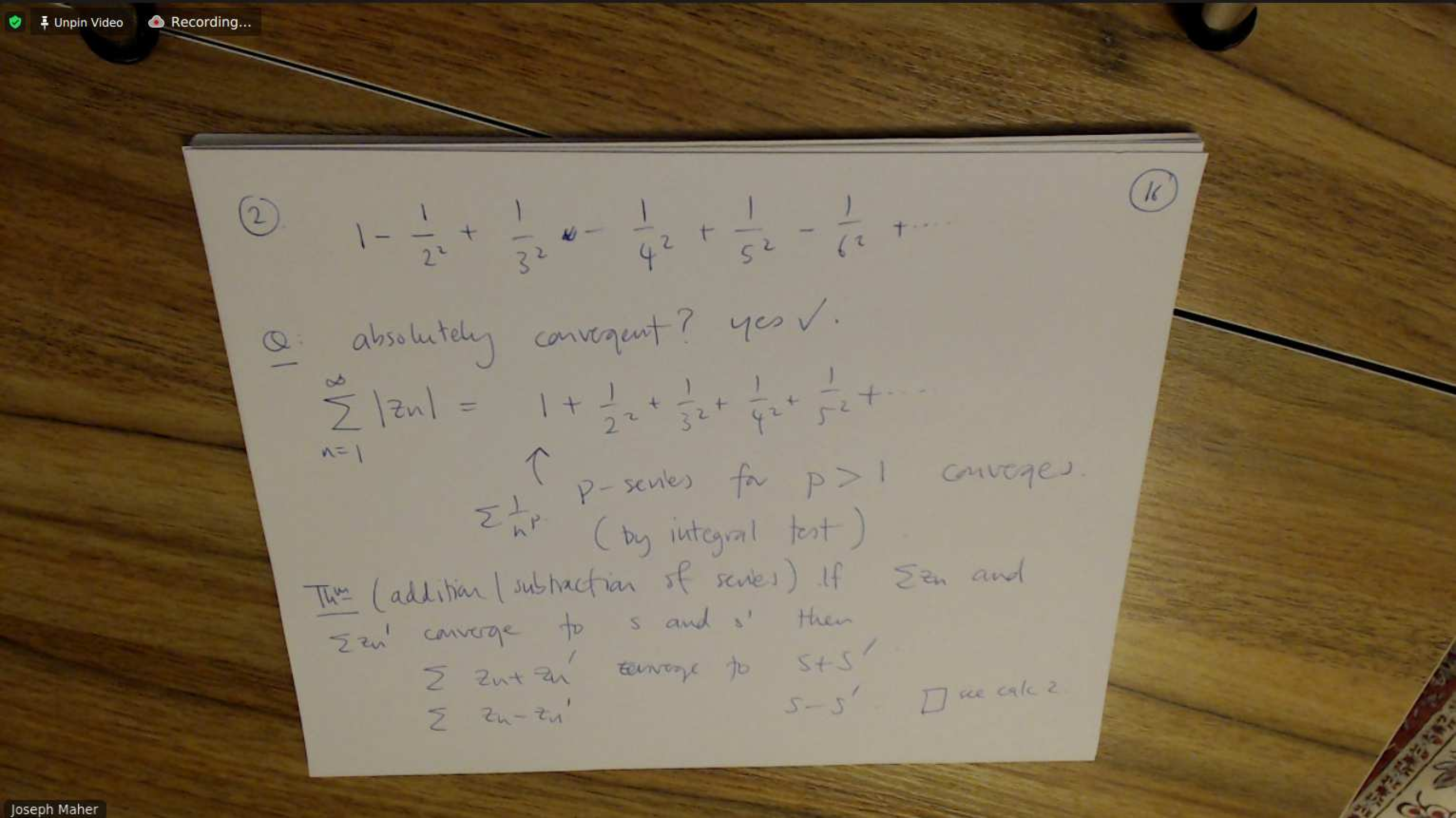
harmonic series, divergent

but (conditionally) convergent

alternating series test (real series).

if a_n is a positive series, a_n decreasing $a_n \to 0$ as $n \to \infty$

then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges



② $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$

Q: absolutely convergent? yes ✓.

$$\sum_{n=1}^{\infty} |z_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

↑

$\sum \frac{1}{n^p}$ p-series for $p > 1$ converges.

(by integral test)

Thm (addition / subtraction of series) If $\sum z_n$ and $\sum z_n'$ converge to s and s' then

$\sum z_n + z_n'$ converge to $s + s'$

$\sum z_n - z_n'$ $s - s'$ ☐ see calc 2.