1

Chap 4 Q19 a) $f: z \mapsto 2z+1-3i$

$f: \mathbb{C} \to \mathbb{C}$

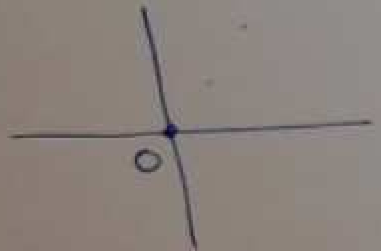
expansion by 2, then add $1-3i$

Q: is there a fixed point? $z = f(z)$

$$z = 2z + 1 - 3i$$

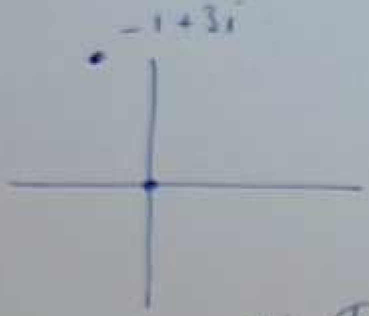
$$-z = 1 - 3i \qquad z = -1+3i$$

$\mathbb{C}$ $z \mapsto 2z$



$h: z \mapsto 2z$

$\underbrace{g h g^{-1}}_{f}(z)$



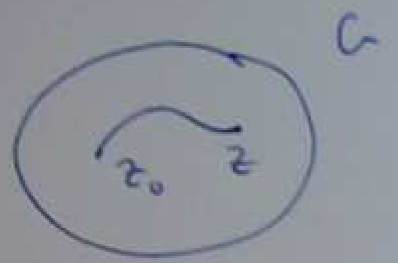
$g: \mathbb{C} \to \mathbb{C}$

$z \mapsto -1+3i$

$g^{-1}\ z \mapsto 1 \mp 3i$

$$z \xrightarrow{g^{-1}} z+1-3i \xrightarrow{h} 2z+2-6i \mapsto 2z+1-3i.$$

last time:



$$F(z) = \int_{z_0}^{z} f(w)\,dw.$$

Def^{n} A function $F(z)$ s.t. $F'(z) = f(z)$ in a domain $G \subseteq \mathbb{C}$ is called an indefinite integral (a antiderivative) for f.

Th^{m} Every indefinite integral of $f(z)$ has the form

$$H(z) = F(z) + c = \int_{z_0}^{z} f(w)\,dw + c \qquad c \text{ const} \in \mathbb{C}.$$

Proof suppose F, H are two indefinite integrals for f.

$$F' - H' = f - f = 0 \qquad F - H = u + iv$$

$$\frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial x} = 0$$

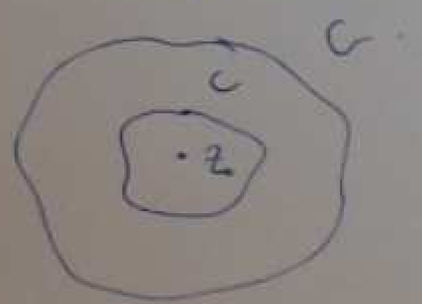
$$\frac{\partial v}{\partial y} = 0 \quad \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow F - H = \text{constant} \quad \square$$

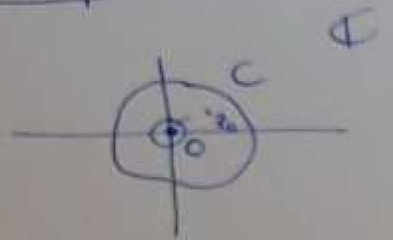
§5.6 Cauchy's integral formula

Thm $f(z)$ analytic ~~fcn~~ in a domain G containing a piecewise smooth Jordan curve, and interior of curve is contained in G. Then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad \text{if } z_0 \text{ lies in int } C$$
$$0 \quad \text{otherwise.}$$



Example (last time) $\int_C \frac{1}{z}\,dz = 2\pi i$



$f(z) = 1$
$z_0 = 0$

$$\frac{1}{2\pi i}\int_C \frac{1}{z}\,dz = 1$$

$$\frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\,dz = \begin{cases} \cancel{2\pi i}\,1 & z_0 \in \text{int}(C) \\ 0 & \text{otherwise } z_0 \notin \text{int}(C) \end{cases}$$

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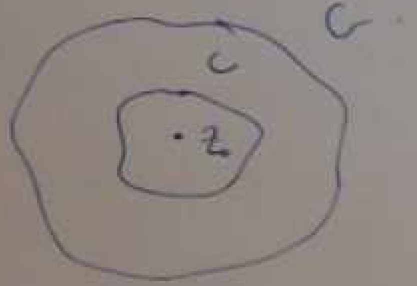

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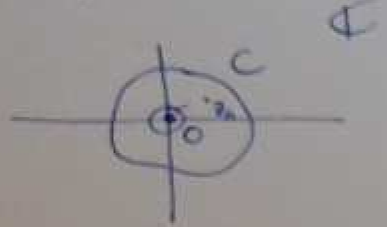


$f(z) = 1$

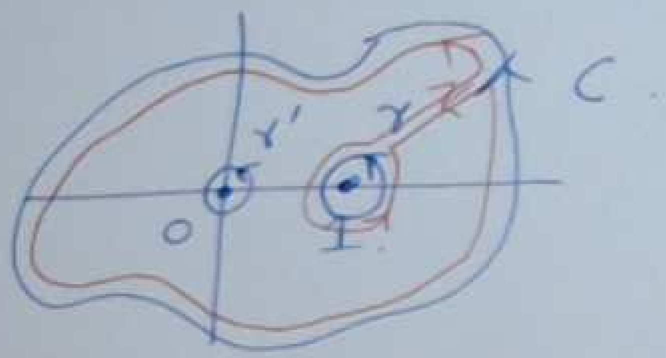
$z_0 = 0$

$$\frac{1}{2\pi i}\int_C \frac{1}{z}\,dz = 1$$

$$\frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\,dz = \begin{cases} \cancel{2\pi i}\,1 & z_0 \in int(C) \\ 0 & \text{otherwise } z_0 \notin int(C) \end{cases}$$

$f(z) = z^2$ $\qquad \frac{1}{2\pi i}\int_C \frac{z^2}{z-\cancel{z_0}\,1}\, dz.$

$f(1) = 1.$



$f(z)$ analytic in G.

$\frac{f(z)}{z-z_0}$ analytic in $G \setminus \{z_0\}$.

substitution / change of coords

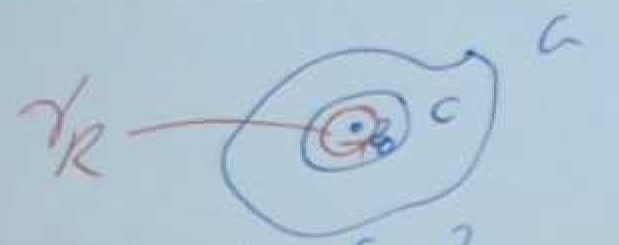
$w = z-1$.

$\frac{dw}{dz} = 1$.

$$\frac{1}{2\pi i}\int_C \frac{z^2}{z-1}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{z^2}{z-1}\, dz$$

$$= \frac{1}{2\pi i}\int_{\gamma'} \frac{(w+1)^2}{w}\, d\cancel{z}w \qquad = \frac{1}{2\pi i}\underbrace{\int_{\gamma'} \frac{1}{w}\, dw}_{2\pi i} = 1.$$

$$= \frac{1}{2\pi i}\int_{\gamma'} \frac{w^2+2w+1}{w}\, dw = \frac{1}{2\pi i}\int_{\gamma'} \underbrace{w+2}_{0} + \frac{1}{w}\, dw.$$



Thm $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0)$.

Proof $z_0 \in \mathrm{int}(c)$, so $\dfrac{f(z)}{z-z_0}$ is analytic in $\mathrm{int}(c)\setminus\{z_0\}$.

Let γ_R be a circle of radius R about z_0, R small, contained in $\mathrm{int}(C)$

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz \quad \leftarrow \text{doesn't depend on } R!$$

$$= \lim_{R\to 0}\int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz$$

claim $\displaystyle\lim_{R\to 0}\int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z)$

want to show for any $\epsilon>0$ there is a $\delta>0$ s.t.

$$\left|\int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)\right| < \epsilon \quad \text{for all } R<\delta$$

$$\left| \int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz - 2\pi i\, f(z_0) \right| = \left| \int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz - f(z_0)\int_{\gamma_R} \frac{1}{z-z_0}\,dz \right|$$

$$= \left| \int_{\gamma_R} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \quad (*)$$

$f(z)$ is cts at z_0, so for all $\frac{\epsilon}{2\pi} > 0$ there is a $\delta > 0$ s.t. $|z-z_0| < \delta$ then $|f(z)-f(z_0)| < \frac{\epsilon}{2\pi}$

$(*)$ FACT $\left| \int_{\gamma_R} g(z)\,dz \right| \leq \int_{\gamma_R} |g(z)|\,dz \leq \max|g(z)| \times \text{length}(\gamma_R)$

$$(*) \leq \int_{\gamma_R} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| dz \leq 2\pi R \,\frac{\epsilon}{2\pi}\, \frac{1}{R} = \epsilon$$

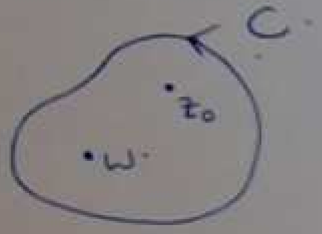
$$\Rightarrow \int_{\gamma_R} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \quad \square$$

Remarks

① if $z_0 \notin \text{int}(C)$ $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz = 0$.

$$\boxed{0! = 1}$$

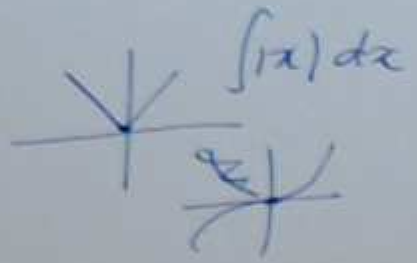
② $f(z)$ analytic on C and int C then values on C determine the values of f in the int(C)



$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \leftarrow$$

Cauchy's integral theorem.

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$



§5.7 Infinite differentiability

Thm If $f(z)$ is analytic in a domain G, then $f(z)$ is infinitely differentiable in G, i.e. $f(z)$ has derivatives $f^{(n)}(z)$ of all orders

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

$z_0 \in \text{int}\,C$

furthermore

(C is a piecewise smooth Jordan curve $\oint$, contained in G, $z_0 \in \operatorname{int}(C)$) (8)

<u>Thm</u> $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad (*)$$

<u>Proof</u> (induction on n) base $n=0$ ✓ $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$.

assume true for $n-1$

consider $$f^{(n)}(z_0) = \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h}$$

— can replace $f^{(n-1)}(-)$ by $\int_C$ (*)

— can replace $\int_C$ by $\int_{\gamma_r}$ ← small circle around z_0.

$$f^{(n)}(z_0) = \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h}$$

assume $h << R$.

$$= \lim_{h\to 0} \frac{(n-1)!}{2\pi i h}\int_{\gamma_R} \frac{f(z)}{(z-z_0-h)^n} - \frac{f(z)}{(z-z_0)^n}\,dz$$

$$= \frac{(n-1)!}{2\pi i h}\int_{\gamma_R} f(z)\,\frac{(z-z_0)^n-(z-z_0-h)^n}{(z-z_0-h)^n(z-z_0)^n}\,dz$$

Fact $\quad a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\dots+b^{n-1})$

$$= \frac{(n-1)!}{2\pi i \cancel{h}}\int_{\gamma_R} f(z)\,\frac{\cancel{h}\left((z-z_0)^{n-1}+(z-z_0)^{n-2}(z-z_0-h)+\dots+(z-z_0-h)^{n-1}\right)}{(z-z_0-h)^n(z-z_0)^n}\,dz$$

$$= \frac{(n-1)!}{2\pi i}\int_{\gamma_R} f(z)\, \frac{(z-z_0)^{n-1} + (z-z_0)^{n-2}(z-z_0-h) + \cdots + (z-z_0-h)^{n-1}}{(z-z_0-h)^n\,(z-z_0)^n}\, dz$$

consider $\left| \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} - \frac{n!}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$

$$= \left| \frac{(n-1)!}{2\pi i}\int_{\gamma_R} f(z)\, \frac{(z-z_0)^{n} + \cdots + (z-z_0-h)^{n-1}(z-z_0) - n(z-z_0-h)^n}{(z-z_0-h)^n\,(z-z_0)^{n+1}}\, dz \right|$$

now use $\left|\int_{\gamma_R} f(z)\,dz\right| \le \int_{\gamma_R} |f(z)|\,dz$ set $M = \max_{z\in\gamma_R} |f(z)|$

$$\le \frac{(n-1)!}{2\pi}\, 2\pi R M \cdot \frac{(2R)^n + 2(2R)^{n-1} + \cdots n - n(2R)^n}{(R-|h|)^n\, R^{n+1}} \longrightarrow 0 \text{ as } R \to 0$$

also use $|z-z_0-h| \le |z_0-z| + |h| \le 2R$

$|R-|h|| = ||z-z_0|-|h||$

so $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_{\gamma_R}} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ □

$$= \frac{(n-1)!}{2\pi i}\int_{\gamma_R} f(z) \frac{(z-z_0)^{n-1} + (z-z_0)^{n-2}(z-z_0-h) + \cdots + (z-z_0-h)^{n-1}}{(z-z_0-h)^n (z-z_0)^n}\, dz$$

consider $\left| \dfrac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} - \dfrac{n!}{2\pi i}\displaystyle\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$

$$= \left| \frac{(n-1)!}{2\pi i} \int_{\gamma_R} f(z) \frac{(z-z_0)^{n} + \cdots + (z-z_0)(z-z_0-h)^{n-1} - n(z-z_0-h)^n}{(z-z_0-h)^n (z-z_0)^{n+1}}\, dz \right|$$

now use $\left|\int_{\gamma_R} f(z)\,dz\right| \le \int_{\gamma_R} |f(z)|\,dz$ set $M = \max_{z\in\gamma_R}|f(z)|$

$$\le \frac{(n-1)!}{2\pi}\, 2\pi R M \cdot \frac{(2R)^n + 2(2R)^{n-1} + \cdots n - n(2R)^n}{(R-|h|)^n R^{n+1}} \longrightarrow 0 \text{ as } R \to 0$$

also use $\le |z-z_0-h| \le |z_0-z|+|h| \le 2R$

$|R-|h|| = ||z-z_0|-|h||$ so $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ □

Proof $F(z) = \int_{z_0}^{z} f(w)\,dw$, defines an analytic function on $\mathbb{C}$, with derivative $F'(z) = f(z)$,

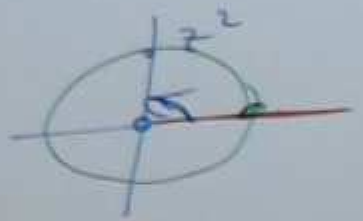
but F analytic $\Rightarrow$ all derivatives are analytic $\Rightarrow f$ analytic. $\square$

eg. $\mathbb{C} \longrightarrow \mathbb{C}$.

Preview: $\mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto z^2$

doubles angles.



$\mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto \exp(z) = e^z$

$z = x+iy$

$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$\cos y + i\sin y$ $\quad = e^x \cos y + i e^x \sin y$