# Remarks HW 2

special case

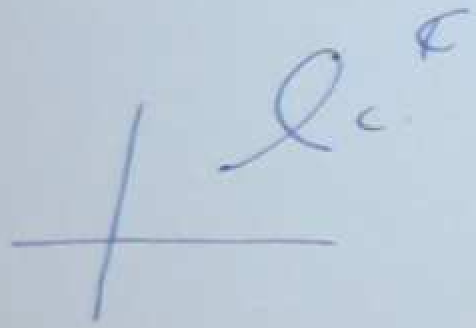
$$
\begin{tikzcd}
c : [a,b] \arrow[r, "cts"'] \arrow[rr, bend left, "cts"] & \mathbb{C} \arrow[r] & \mathbb{R}
\end{tikzcd}
$$

$$z \xmapsto{cts} |z|.$$

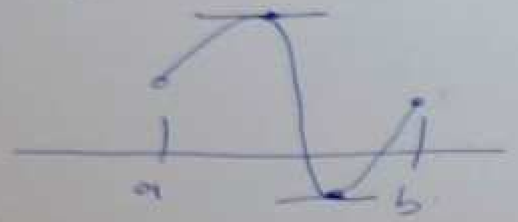
show: bounded.

$$[a,b] \xrightarrow{cts} \mathbb{R}.$$

closed
interval
bounded




<u>Fact</u> (calc / real analysis) a cts function on a closed bounded interval achieves its max/min.



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(2)

$E \subseteq \mathbb{C} \xrightarrow{f} \mathbb{C}$ show $f$ is bounded on $E$.

closed
bounded

suppose not, then there is a sequence

$(z_n) \in E$ s.t. $f(z_n) \longrightarrow \infty$. ( means exactly $|f(z_n)| \to \infty$ )

BW $\Rightarrow$ $(z_n)$ has a limit $z \in E$

<u>Fact</u>: $z$ is a limit point of $(z_n)$.

$\Leftrightarrow$ there is a subsequence $(z_{n_k})$ st $(z_{n_k}) \longrightarrow z \in E$

$f(z_{n_k}) \longrightarrow f(z) \leftarrow$ some finite number

$\#$ $f(z_{n_k}) \to \infty$.

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$f(z) = \frac{1}{1-z}$ $\quad |z| < 1$. $\quad$ cts.

- compositions of cts functions are cts.

$f'(z) = \frac{1}{(1-z)^2}$

- not uniformly cts on $|z| < 1$.

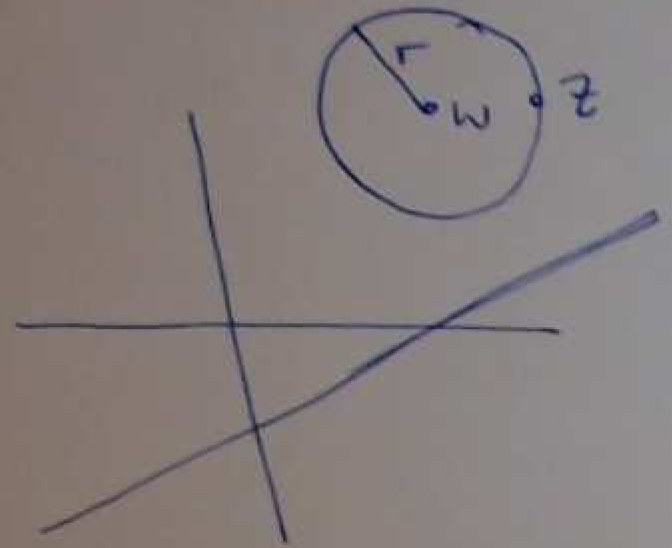
<u>unif cts:</u> for all $\epsilon > 0$, there is a $\delta > 0$ s.t. for all $z_1, z_2$

$|z_1 - z_2| < \delta$, $\quad |f(z_1) - f(z_2)| < \epsilon$.

$$\frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} < \frac{\epsilon}{\delta}$$ finite number.

$\Rightarrow$ $f'(z)$ bounded.

$\mathbb{C}$




$$|z-w|^2 = r^2$$

$$(z-w)\overline{(z-w)} = r^2.$$

$$(z-w)(\bar z - \bar w) = r^2.$$

$$z\bar z - \bar w z - w \bar z + w\bar w = r^2.$$

$B = -A\bar w$

$-\frac{B}{A} = \bar w$

$\frac{|B|^2}{A^2} = |w|^2$

$\boxed{A \in \mathbb{R}_+}$

$$A|z|^2 - \underbrace{A\bar w}_{B} z - Aw\bar z + \underbrace{Aw\bar w - Ar^2}_{\in \mathbb{R}} = 0.$$

$B = A\bar w$
$\in \mathbb{C}.$

$C \in \mathbb{R}$

$$\boxed{A|z|^2 + Bz + \bar B \bar z + C = 0}$$

$C = A(|w|^2 - r^2)$

$$C = A\left(\frac{|B|^2}{A^2} - r^2\right).$$

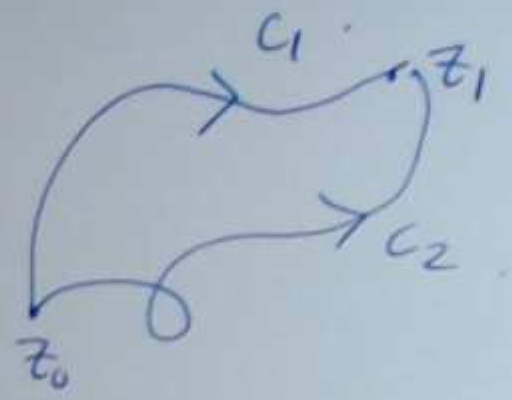
$Ar^2 = |B|^2 - AC.$

want $r > 0.$

$r > 0 \Leftrightarrow |B|^2 > AC.$

$AC = |B|^2 - Ar^2$

# § Cauchy's integral theorem




$\mathbb{C}$.

If $c_1, c_2$ have the same endpoints and

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

then $\displaystyle\int_{c_1 \cup \bar{c}_2} f(z)\,dz = 0$.

so if integrals are path independent

$\iff \displaystyle\int_{\text{closed loops}} f(z)\,dz = 0$.


$\int_c f(z)\,dz = 0$.

Q: when does this happen?

- know it happens for polynomials.

**Thm** (Cauchy's integral theorem) Let $f(z)$ be analytic in a simply connected domain $G$. Then $\int_C f(z)\,dz = 0$ for all piecewise smooth curves $C$ contained in $G$.


$\int_C f(z)dz = 0.$

**Proof** claim: it suffices to show this for all triangles $\Delta \subseteq G$ $\int_\Delta f(z)\,dz = 0$.

**Proof** (CIT) assuming claim.

The piecewise smooth curve $C$ can be approximated by polygonal curve $L$, $\left| \int_C f(z)\,dz - \int_L f(z)\,dz \right| < \epsilon$.


$\epsilon$

Then $\int_L f(z)\,dz$ can be replaced by $\sum_i \int_{\Delta_i} f(z)\,dz$. $(*)$

sum of integrals around triangles.

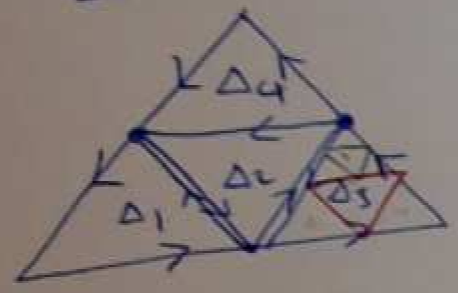

claim $\Rightarrow$ $(*) = 0$ $\quad \left| \int_C f(z)\,dz \right| < \epsilon$

$\Rightarrow \int_C f(z)\,dz = 0$. $\square$

<u>Proof</u> (of claim) $\Delta$ triangle in $G$ (for a contradiction)

assume $\left| \int_\Delta f(z)\,dz \right| = M > 0$.

$\Delta$




subdivide $\Delta$ into 4 congruent triangles

$$M = \left| \int_\Delta f(z)\,dz \right| = \left| \int_{\Delta_1} + \int_{\Delta_2} + \int_{\Delta_3} + \int_{\Delta_4} \right| = M.$$

at least one $\Delta_i$ has $\left| \int_{\Delta_i} f(z)\,dz \right| \geq \frac{M}{4}$.

call this $\Delta^{(1)}$

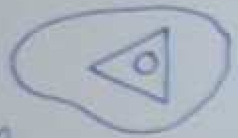
now subdivide this one, call this new one $\Delta^{(2)}$

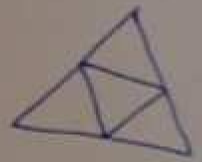
$$\left| \int_{\Delta^{(2)}} f(z)\,dz \right| \geq \frac{M}{4^2}.$$

carry on subdividing in this way

get a sequence of nested triangles

$\Delta \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \Delta^{(3)} \supset \cdots$ s.t. $\left| \int_{\Delta^{(k)}} f(z)\,dz \right| \geq \frac{M}{4^k}$.

note this uses $G$ simply connected.  ← doesn't work for this.

note  if the perimeter of $\Delta$ is $\ell$. then the perimeter of $\Delta^{(k)}$ is $\frac{\ell}{2^k}$.

claim nested rectangles argument works for nested triangles. $\square$.


so there is a unique point $z_0 = \bigcap_k \Delta^{(k)}$.

$f$ analytic $\Rightarrow$ $f'(z_0)$ is finite complex number.

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for all $\epsilon > 0$, there is a $\delta > 0$ s.t. if $|z - z_0| < \delta$

then $\left| \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$.

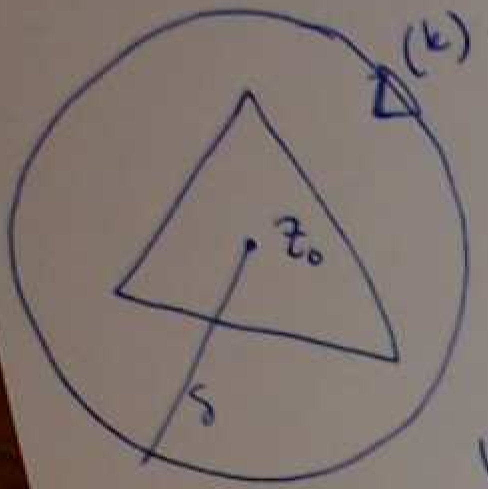
$\Leftrightarrow \quad | f(z) - f(z_0) - f'(z_0)(z - z_0) | < \epsilon |z - z_0|$.

$$\int_{\Delta^{(k)}} f(z) - f(z_0) - (z - z_0) f'(z_0) \, dz$$

$$= \int_{\Delta^{(k)}} f(z)\, dz - f(z_0) \underbrace{\cancel{\int_{\Delta^{(k)}} 1 \, dz}}_{0} - f'(z_0) \overbrace{\cancel{\int_{\Delta^{(k)}} z \, dz}}^{0}$$

$$+ z_0 f'(z_0) \underbrace{\cancel{\int_{\Delta^{(k)}} 1 \, dz}}_{0}.$$

Know Cauchy's Thm holds for polynomials.




$\Delta^{(k)} \subset |z-z_0|<\delta$.

set $\ell_k =$ perimeter of $\Delta^{(k)}$

$= \dfrac{\ell}{2^k}$.

~~for $k$ sufficiently~~

we can choose $\delta$ small enough s.t.

$|z-z_0| < \ell_k$.

Fact

$\left|\int_C f(z)\,dz\right|$

$< \int_C |f(z)|\,dz$

$< \max|f(z)|$ length$(C)$

recall $|f(z)-f(z_0)-f'(z_0)(z-z_0)| < \epsilon\,|z-z_0|$

$< \epsilon\,\ell_k$

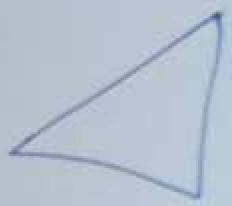
$$\left|\int_{\Delta^{(k)}} f(z)\,dz\right| = \left|\int_{\Delta^{(k)}} f(z)-f(z_0)-f'(z_0)(z-z_0)\,dz\right|$$

$< |f(z)-f(z_0)-f'(z_0)(z-z_0)|$ length of $\Delta^{(k)}$.

$$< \epsilon\,\underset{\frac{\ell}{2^k}}{\underset{\shortparallel}{\ell_k}}\;\underset{\frac{\ell}{2^k}}{\underset{\shortparallel}{\ell_k}} = \epsilon\,\frac{\ell^2}{4^k}.$$

just did

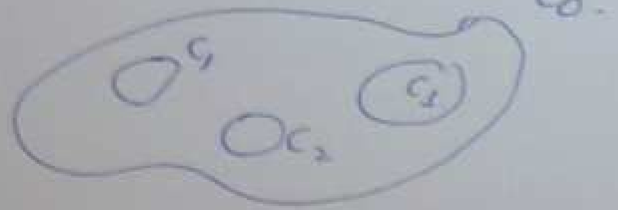
$$\frac{M}{4^k} \le \left| \int_{\Delta^{(k)}} f(z)\,dz \right| \le \epsilon \frac{\ell^2}{4^k}.$$


$\Delta$.

holds for all $\epsilon > 0$. $M \le \epsilon \ell^2$.

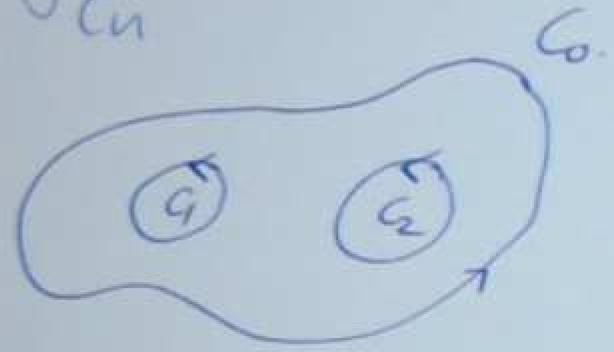
$\Rightarrow M = 0$. # $M > 0$. $\square$.

<u>Multiple curves</u> $C_0$ piecewise smooth Jordan curve containing $c_1, c_2, \ldots c_n$ piecewise smooth Jordan curves with disjoint interiors.

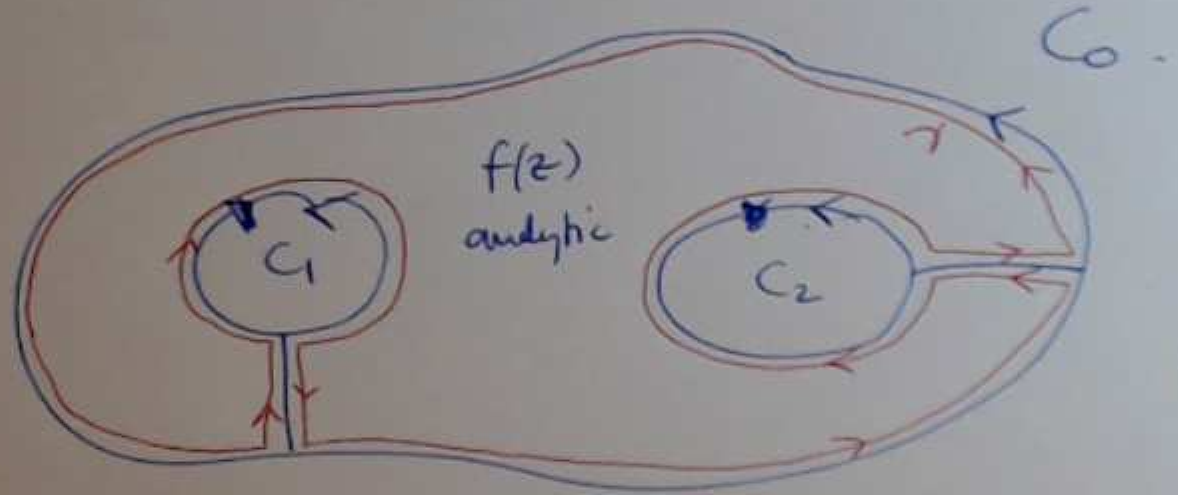



$f(z)$ analytic in $\operatorname{int}(c_0) \setminus \bigcup_{i=1}^{n} \operatorname{int}(c_i)$

**claim** $\int_{C_0} f(z)\,dz = \int_{C_1} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$




**Proof** add dividing arcs. $\square$




$\int_\gamma f(z)\,dz = 0$

by Cauchy's Integral Thm.

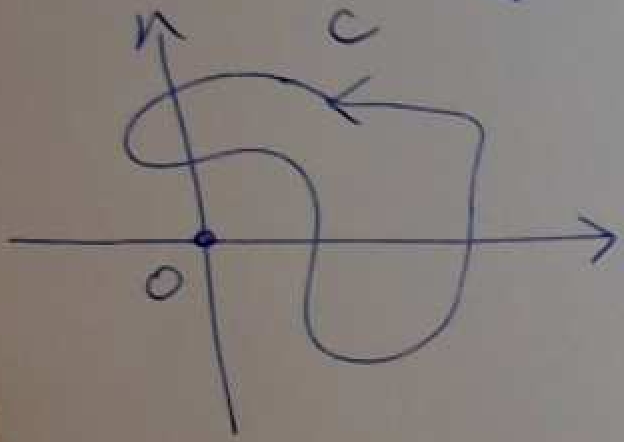
$$\int_\gamma f(z)\,dz = \int_{C_0} f(z)\,dz + \cancel{\int_{arcs} f(z)\,dz} - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\int_{C_0} f(z)\,dz = \int_{C_1} f(z)\,dz + \cdots + \int_{C_2} f(z)\,dz$$

<u>Example</u> $f(z) = \frac{1}{z}$ $\quad f'(z) = -\frac{1}{z^2}$ analytic except at $z=0$.

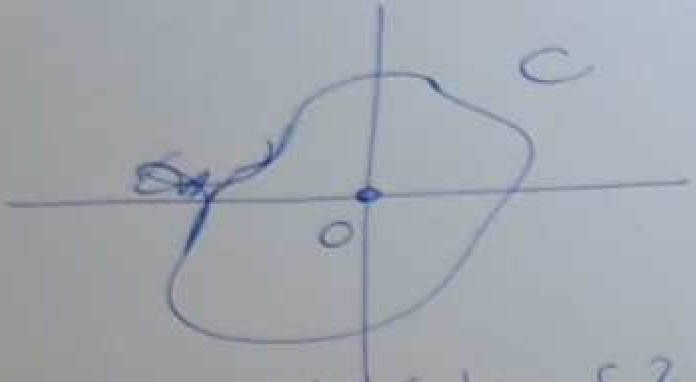
$C$ Jordan curve. $\qquad \int_C \frac{1}{z}\,dz = 0$

$\mathbb{C}$.




as long as $C$ does not contain $0$ in its interior.

$\mathbb{C}$.



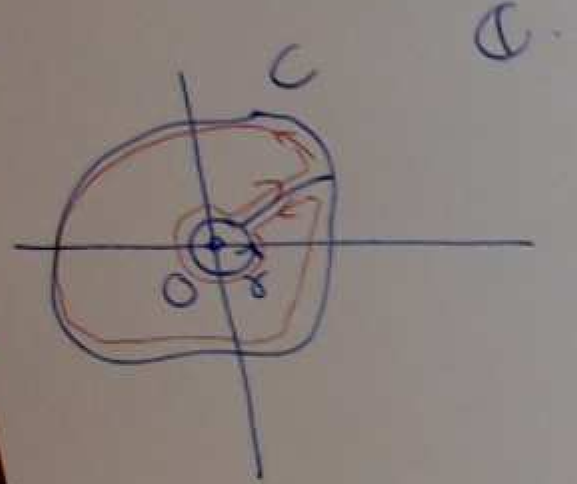

<u>Q</u>: suppose $0 \in \text{int}(C)$ ?

Cauchy's Thm does not apply

$f$ analytic on $\text{int}(C) \setminus \{0\}$

$f(z) = \frac{1}{z}$.

Q: $\int_C \frac{1}{z}\, dz$ ?




claim $= \int_\gamma \frac{1}{z}\, dz$

where $\gamma$ is a circle of radius $R$ sufficiently small s.t. $\gamma \subset \operatorname{int}(C)$

$$\int_C \frac{1}{z}\, dz = \int_\gamma \frac{1}{z}\, dz$$


parameterize $\gamma$: $\gamma(\theta) = R\cos\theta + iR\sin\theta$.

$\gamma'(\theta) = -R\sin\theta + iR\cos\theta$

$$\int_\gamma \frac{1}{z}\, dz$$

$$\gamma(\theta) = R\cos\theta + iR\sin\theta$$

$$\gamma'(\theta) = -R\sin\theta + iR\cos\theta$$

$$\int_0^{2\pi} \frac{1}{R\cos\theta + iR\sin\theta}\, \gamma'(\theta)\, d\theta$$

$$\int_0^{2\pi} \frac{1}{R\cos\theta + iR\sin\theta}\left(-R\sin\theta + iR\cos\theta\right) \frac{R\cos\theta - iR\sin\theta \;\cancel{d\theta}}{R\cos\theta - iR\sin\theta}\, d\theta$$

$$\int_0^{2\pi} \frac{\cancel{R^2}\left(\cancel{-\sin\theta\cos\theta} + \cancel{\sin\theta\cos\theta} + \underbrace{i\cos^2\theta + i\sin^2\theta}_{1}\right)}{\cancel{R^2}\left(\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right)}\, d\theta$$

$$\int_0^{2\pi} i\, d\theta = 2\pi i$$

$$\int_\gamma \frac{1}{z}\,dz$$

$$\gamma(\theta) = R\cos\theta + iR\sin\theta$$

$$\gamma'(\theta) = -R\sin\theta + iR\cos\theta$$

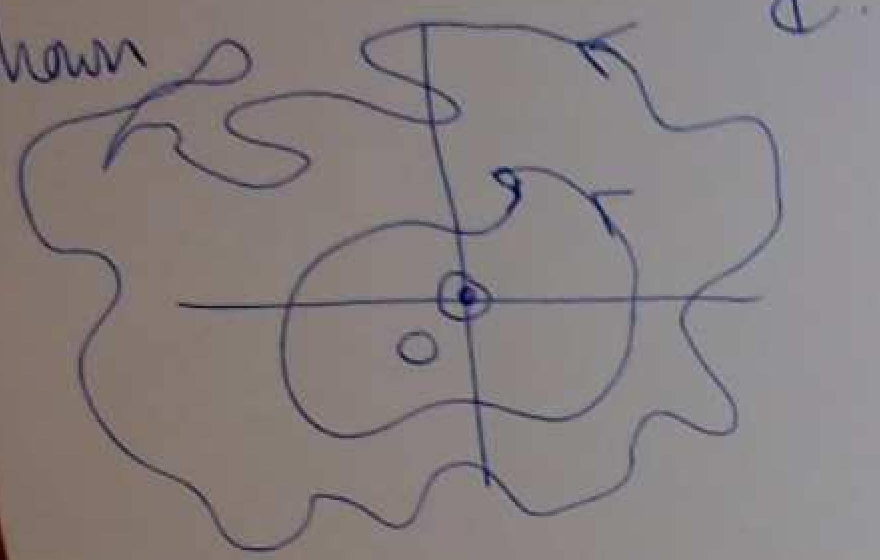
$$\int_0^{2\pi} \frac{1}{R\cos\theta + iR\sin\theta}\,\gamma'(\theta)\,d\theta$$

$$\int_0^{2\pi} \frac{1}{R\cos\theta + iR\sin\theta}\left(-R\sin\theta + iR\cos\theta\right)\frac{R\cos\theta - iR\sin\theta \quad \cancel{d\theta}}{R\cos\theta - iR\sin\theta}\,d\theta$$

$$\int_0^{2\pi} \frac{\cancel{R^2}\left(\cancel{-\sin\theta\cos\theta} + \cancel{\sin\theta\cos\theta}\right.}{\cancel{R^2}\left(\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right)}\left. + \underbrace{i\cos^2\theta + i\sin^2\theta}_{1}\right)\,d\theta$$

$$\int_0^{2\pi} i\,d\theta = 2\pi i$$

shown



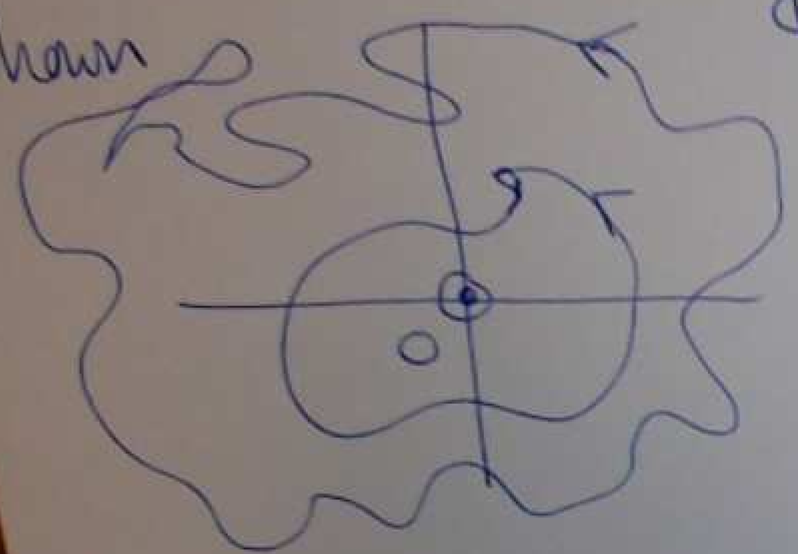

$$\int_C \frac{1}{z}\,dz = \int_\gamma \frac{1}{z}\,dz = 2\pi i.$$

More generally consider $f(z) = \dfrac{1}{z-z_0}$ $\quad \displaystyle\int_C \frac{1}{z-z_0}\,dz$.

If $C$ contains $z_0$ then $\displaystyle\int_C \frac{1}{z-z_0}\,dz = 2\pi i$

if $C$ does not contain $z_0$ then $\displaystyle\int_C \frac{1}{z-z_0}\,dz = 0$.

shown



$\mathbb{C}$

$$\int_C \frac{1}{z}\,dz = \int_\gamma \frac{1}{z}\,dz = 2\pi i$$

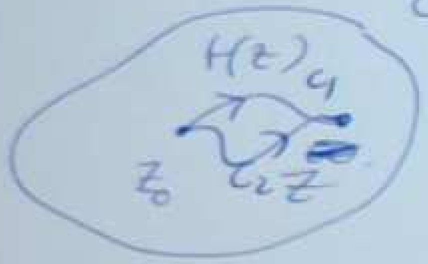
**More generally** consider $f(z) = \dfrac{1}{z-z_0}$ $\quad \displaystyle\int_C \frac{1}{z-z_0}\,dz$

If $C$ contains $z_0$ then $\displaystyle\int_C \frac{1}{z-z_0}\,dz = 2\pi i$

if $C$ does not contain $z_0$ then $\displaystyle\int_C \frac{1}{z-z_0}\,dz = 0$

# Indefinite integrals

**Thm** $f(z)$ analytic in a simply connected domain $C$. Then $F(z) = \int_{z_0}^{z} f(w)\,dw$



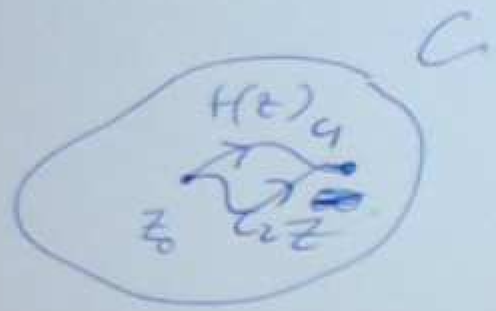

along any piecewise smooth curve from $z_0$ to $z$, and this defines an analytic function in $C$, $F'(z) = f(z)$.

**Proof** well defined, by Cauchy's integral theorem.

$\int_{\text{closed loop}} f(z)\,dz = 0 \iff$ path independence.

# Indefinite integrals

**Thm** $f(z)$ analytic in a simply connected domain $G$. Then $F(z) = \int_{z_0}^{z} f(w)\,dw$

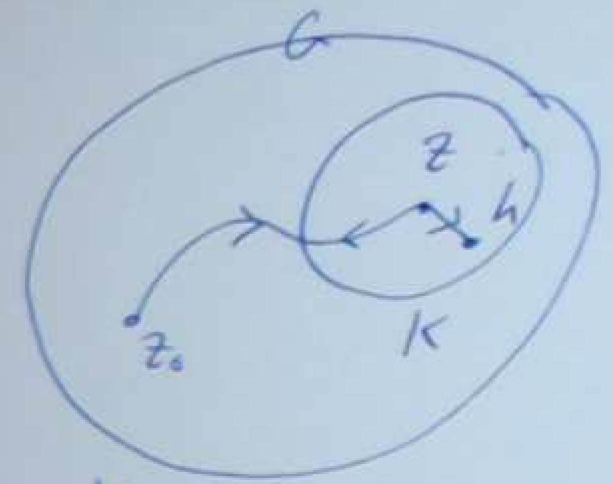



along any piecewise smooth curve from $z_0$ to $z$, and this defines an analytic function in $G$, $F'(z) = f(z)$.

**Proof** well defined, by Cauchy's integral theorem.

$\int_{\text{closed loop}} f(z)\,dz = 0 \iff$ path independence.

## check derivative




choose K to be a neighbourhood of z in G choose h small s.t. $z+h \in K$

$$F(z+h)-F(z) = \int_{z_0}^{z+h} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw = \int_{z}^{h} f(w)\,dw$$

↑ assume straight line path in K

$$\frac{F(z+h)-F(z)}{h} - f(z)$$

$$= \frac{1}{h}\int_{z}^{h} f(w)\,dw - f(z) = \frac{1}{h}\int_{z}^{h} f(w)\,dw - f(z)\frac{1}{h}\int_{z}^{h} 1\,dw$$

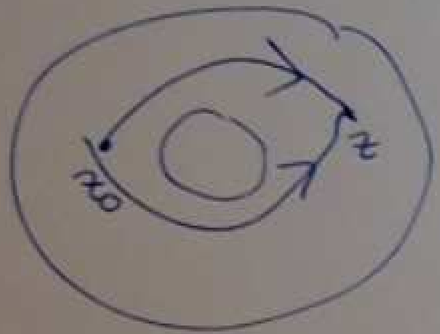
$$= \frac{1}{h}\left|\int_{z}^{h} f(w) - f(z)\,dw\right|$$

$$\leq \frac{1}{h}\cdot \epsilon h = \epsilon \quad \square$$

f cts so for all $\epsilon > 0$ there is a $\delta > 0$ s.t. $|z-w| < \delta$ then $f(w) - f(z) < \epsilon$.

Remarks. ① really only use $f$ cts. for existence of antiderivative.

② what if $G$ not simply connected.




can try and define $F$ but might depend on path.