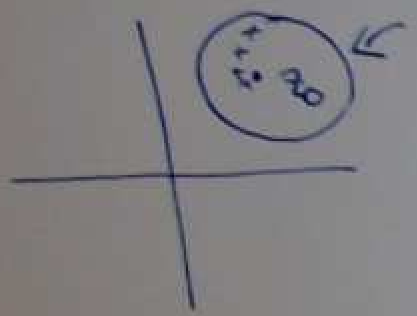
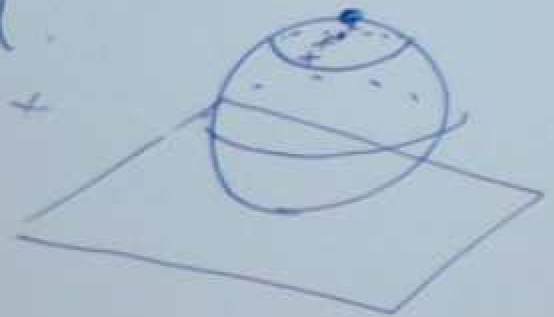
$\mathbb{C}$



$|z-z_0| \le r$ neighborhood

$0 < |z-z_0| \le r$ deleted nbd.

$N = \{\infty\}$



$\mathbb{C}$.

$z \longmapsto \frac{1}{z}$

(z_n)

$|z| \longrightarrow \infty$.

$\Longleftrightarrow \frac{1}{|z_n|} \longrightarrow 0$.

$|z| \ge r$
deleted nbd of $\{\infty\}$

$\lim_{z\to\infty} f(z) = L \iff \lim_{z\to 0} f(\frac{1}{z}) = L$

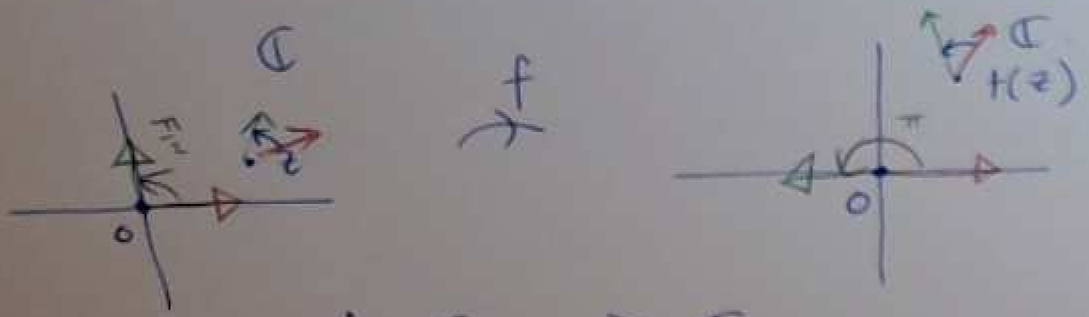
<u>last time</u> <u>Thm</u> $f : \mathbb{C} \longrightarrow \mathbb{C}$ is complex differentiable
analytic

Then f is conformal where $f'(z) \neq 0$.

<u>Example</u> $f : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^2$

$f'(z) = 2z$

$f'(z) \neq 0$ as long as $z \neq 0$.

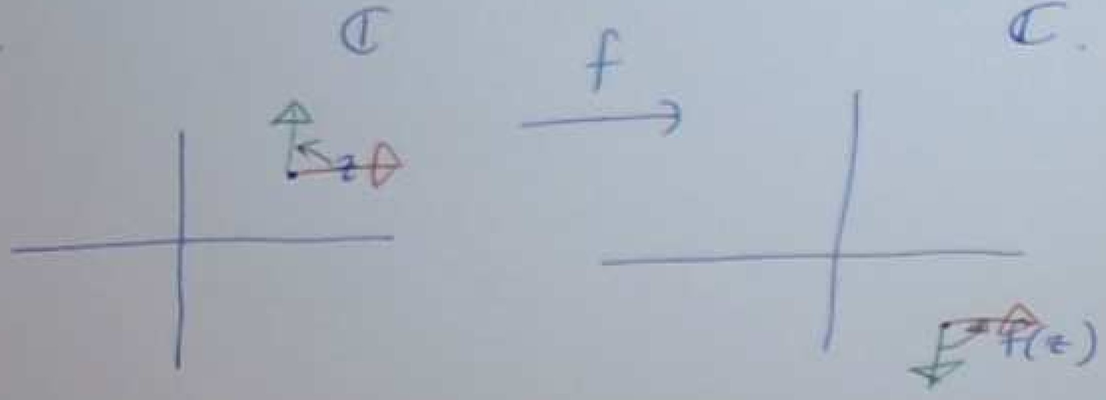


$$\begin{aligned} f : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z^2 \end{aligned}$$

this is conformal on $\mathbb{C} \setminus \{0\}$.

<u>Def$^{\underline{n}}$</u> If $f : \mathbb{C} \longrightarrow \mathbb{C}$ which preserves angles, but <u>not</u> orientations, it's called <u>isogonal</u>.

<u>Example</u> $f : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto \bar{z}$



this is <u>isogonal</u> (angle preserving)

but <u>not</u> conformal

<u>Remark</u> $f'(z)$ linear map $\mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto f'(z)z$

$\arg(f'(z))$ angle of rotation $\quad |f'(z)|$ expansion/contraction factor (scaling) compare with $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

any old 2×2 matrix.

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

doesn't preserve angles.

§5 Integration (along curves)

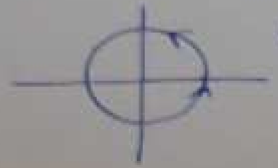
Let $I \subseteq \mathbb{R}$ and let c be a smooth curve

$$c : I \longrightarrow \mathbb{C}.$$
$$t \longmapsto z(t)$$
$$c(t)$$

Example

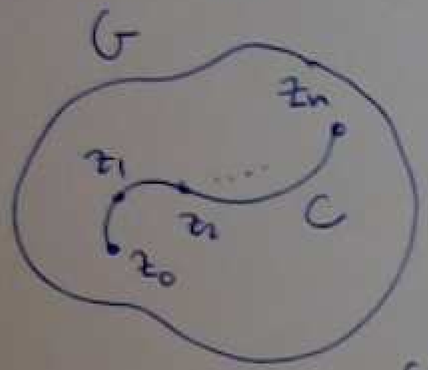
$$t \longmapsto \cos t + i \sin t.$$

$0 \leq t \leq 2\pi$



$\mathbb{C}$.

given $f : \mathbb{C} \to \mathbb{C}$ we can integrate it along C.

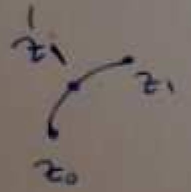


<u>notation</u> $\int_C f(z)\,dz$.

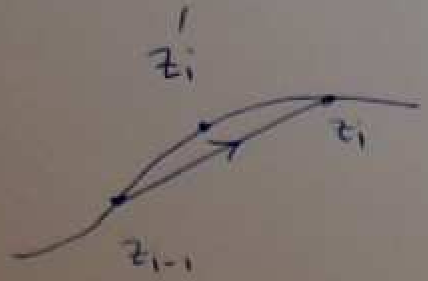
<u>recall</u> Riemann sums.

choose a partion $z_0, z_1, \ldots, z_n$ along C

take $\displaystyle\sum_{i=1}^{n} f(z_i')\,\Delta z_i$ $\qquad$ z_i' a point on the arc from $c(z_{i-1})$ to $c(z_i)$.

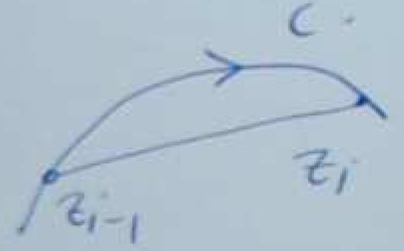


$\Delta z_i = z_i - z_{i-1}$.



set l_i = length of the arc of C

from z_{i-1} to z_i



$$\lambda = \max_i l_i$$

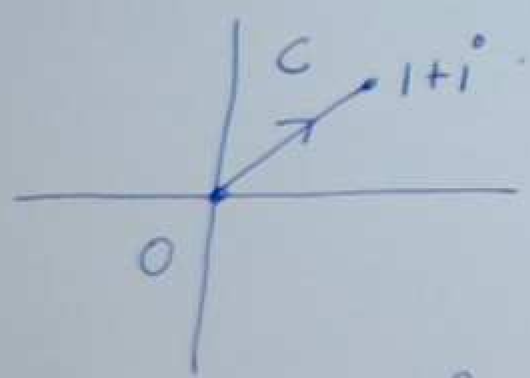
suppose $\displaystyle\lim_{\lambda \to 0} \sum_{i=1}^{n} f(z_i)(z_i - z_{i-1})$ exists.

then we say that f is integrable along C

and $$\int_C f(z)\,dz = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(z_i)(z_i - z_{i-1})$$

Example $f : \mathbb{C} \to \mathbb{C}$

$z \longmapsto a+bi$



$c(t) = t + ti \qquad c'(t) = 1+i$

$0 \leq t \leq 1$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

c

$$\int_c f(z)\,dz = \int_0^1 (a+bi)\,c'(t)\,dt$$

$$= \int_0^1 (a+bi)(1+i)\,dt = \int_0^1 a-b+i(b+a)\,dt$$

$$\Big[(a-b+i(a+b))t\Big]_0^1 = (a-b)+i(a+b)$$

$$\|$$

preview $\displaystyle\int_0^{1+i} (a+bi)\,dz = \Big[(a+bi)z\Big]_0^{1+i} = (a+bi)(1+i)$

Thm If $f(z)$ is continuous in a domain $G \subseteq \mathbb{C}$ containing a smooth curve C, then $f(z)$ is integrable along C.

Proof $z_i = x_i + iy_i \qquad z_i' = x_i' + y_i' i \qquad \Delta z_i = \Delta x_i + i\Delta y_i$

$$\underset{x+iy}{f(z)} = u(x,y) + iv(x,y) \quad \text{then} \quad \begin{matrix} u_i = u(x_i', y_i') \\ v_i = v(x_i', y_i') \end{matrix}$$

$$\sum_i f(z_i')\Delta z_i = \sum_i (u_i + iv_i)(\Delta x_i + i\Delta y_i)$$

$$= \sum_i u_i \Delta x_i - v_i \Delta y_i + i\sum_i (v_i \Delta x_i + u_i \Delta y_i) \quad \text{as } \lambda \to 0$$

$$\int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \qquad \square$$

$$\parallel$$

Remarks ①

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$u(z) + iv(z)$$

② $C : [a,b] \longrightarrow \mathbb{C}$

$t \longmapsto z(t) = x(t) + iy(t)$

$$f(z) = u(x,y) + iv(x,y)$$

$$\int_C f(z)\,dz = \int_C f(z(t))\,z'(t)\,dt$$

$$= \int_C \big(u(z(t)) + iv(z(t))\big)\big(x'(t) + iy'(t)\big)\,dt$$

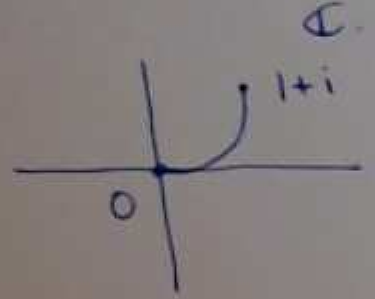
$$= \int_C u(z(t))x'(t) - v(z(t))y'(t)\,dt + i\int_C v(z(t))\,x'(t) + u(z(t))\,y'(t)\,dt$$

$$= \int_C R(t)\,dt + i\int_C I(t)\,dt \quad \text{where}$$

$$R(t) = \mathrm{Re}\left(f(z(t))\, z'(t) \right)$$
$$I(t) = \mathrm{Im}\left(f(z(t))\, z'(t) \right).$$

<u>Example</u> $f(z) = z$

$$\overset{z(t)}{C(t)} = \underset{t+it^2}{(t, t^3)} \qquad 0 \le t \le 1$$



$$f(x+iy) = x+iy \qquad c'(t) = 1 + i3t^2$$

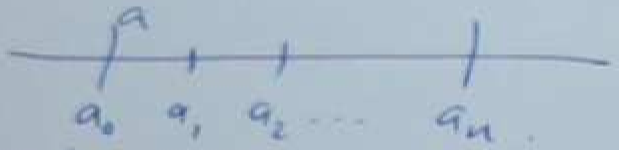
$$f(c(t)) = t + it^3$$

$$f(c(t)) . c'(t) = (t+it^3)(1+i3t^2)$$
$$= \underbrace{t - 3t^5} + i\underbrace{(t^3 + 3t^3)}$$

$$\int_0^1 t - 3t^5\,dt + i\int_0^1 t^3 + 3t^3\,dt$$

Def$^{\underline{n}}$ A curve $C : I \longrightarrow \mathbb{C}$ is <u>piecewise smooth</u>

if there is a subdivision $I = [a,b]$.

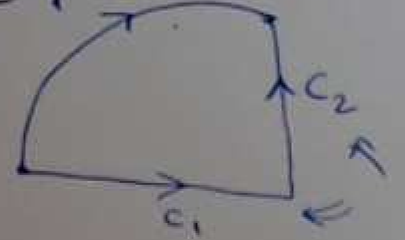


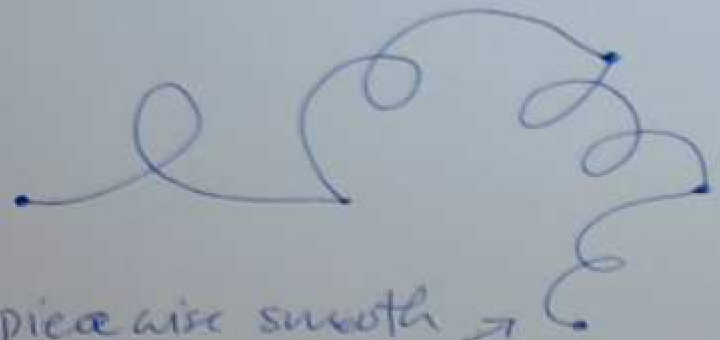
s.t. $C_i = C\big|_{[a_{i-1}, a_i]}$ is a smooth curve.

and C is continuous.

$C \notin$ smooth

Example

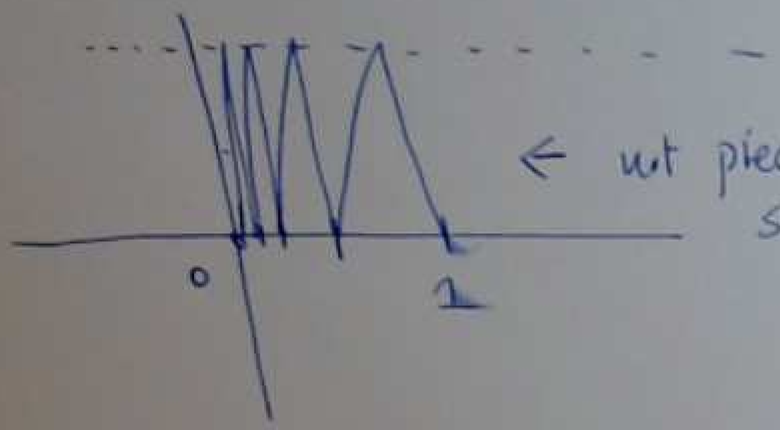




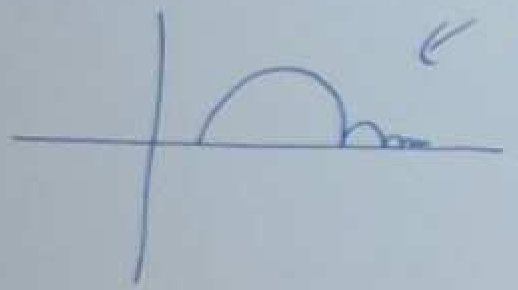
$C_1 \cup C_2$ piecewise smooth

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

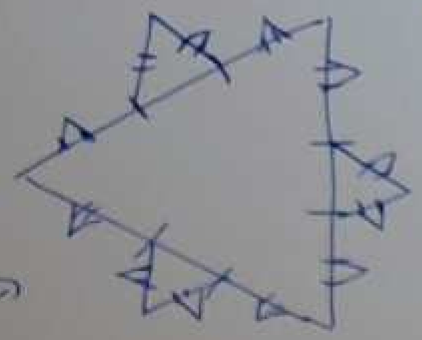
finite number of pieces!



← not piecewise smooth



← not p.s.



← Koch snowflake

← do this infinitely often, get continuous curve

but not piecewise smooth

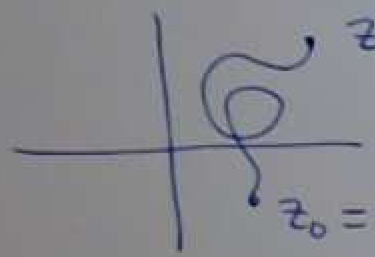
fractal.

(13)

piecewise smooth $\leftrightarrow$ finitely many smooth segments.

Example

$\mathbb{C}$.



$z_1 = c(b)$ C is a piecewise smooth curve from z_0 to z_1.

$z_0 = c(a)$

$$\int_C z^n \, dz = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$

as long as $n \neq -1$ and C does not hit 0 for $n < 0$.

Remark: this does not depend on the path C.

check

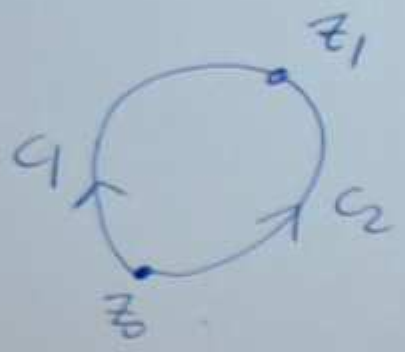
$$\int_C z^n \, dz = \int_a^b z^n(t) \, z'(t) \, dt$$

$$= \int_a^b \frac{1}{n+1} \frac{d}{dt}\left(z^{n+1}(t)\right) dt = \frac{1}{n+1}\left(z^{n+1}(b) - z^{n+1}(a)\right)$$

$$= \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$

Remark if $\int_C f(z)\,dz$ are path independent

$\Longleftrightarrow$ same as $\int_{\text{loops}} f(z)\,dz = 0$



$c = c_1 \cup \overline{c_2}$

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz - \int_{c_2} f(z)\,dz = 0$$

equal by path independence

Notation $\overline{C} \leftarrow$ go around C in the opposite direction

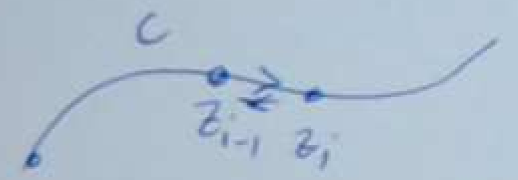
$c : I \longrightarrow \mathbb{C}$ $\qquad$ $\overline{c} : I \longrightarrow \mathbb{C}$

$[a,b] \longrightarrow \mathbb{C}$ $\qquad$ $[a,b] \longmapsto \mathbb{C}$

$t \longmapsto c(t)$ $\qquad$ $t \longmapsto c(a+b-t)$

c^-

useful properties



Thm $\int_C f(z)\,dz = -\int_{\bar{C}} f(z)\,dz$.

$C: \sum f(z_i')(z_i - z_{i-1})$

$\bar{C}: \sum f(z_i)(z_{i-1} - z_i)$

Thm $f(z)$ is continuous on C and $|f(z)| \leq M$.

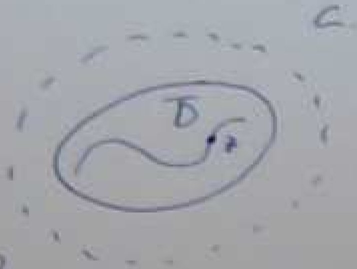
then $\left| \int_C f(z)\,dz \right| \leq M\ell$, ℓ = length of C. $\square$

Approximation C by polygonal curves L

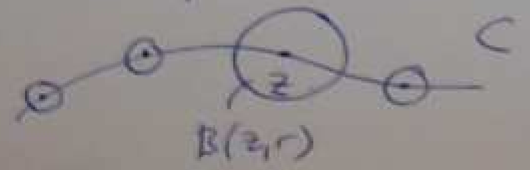
Lemma Let $f(z)$ be continuous on a domain G, C piecewise smooth curve in G. For any $\epsilon > 0$, there is an polygonal curve L determined by C such that

$$\left| \int_C f(z)dz - \int_L f(z)dz \right| < \epsilon .$$

Proof (sketch) **claim** there is a bounded domain D s.t. $C \subseteq D \subseteq G$ s.t. $\overline{D} \subseteq G$.



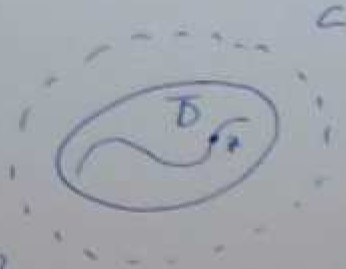
proof (of claim) for each point $z \in C$ there is a neighborhood $B(z,r) = \{ w \mid |w-z| < r \}$, $r > 0$, s.t. $z \in B(z,r) \subseteq G$



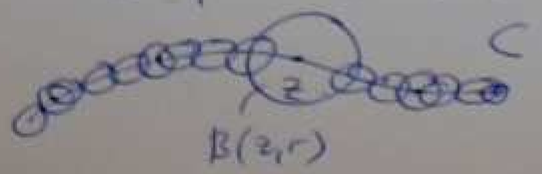
<u>Lemma</u> Let $f(z)$ be continuous on a domain G, C piecewise smooth curve in G. For any $\epsilon > 0$, there is an polygonal curve L determined by C such that

$$\left| \int_C f(z)\,dz - \int_L f(z)\,dz \right| < \epsilon .$$

<u>Proof</u> (sketch) <u>claim</u> there is a bounded domain D s.t. $C \subseteq D \subseteq G$ s.t. $\overline{D} \subseteq G$.



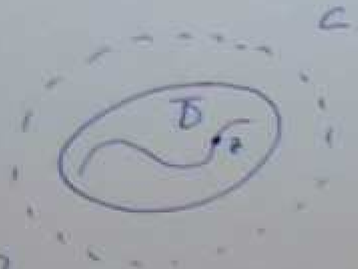
<u>proof</u> (of claim) for each point $z \in C$ there is a neighborhood $B(z,r) = \{ w \mid |w - z| < r \}$ $r>0$ s.t. $z \in B(z,r) \subseteq G$



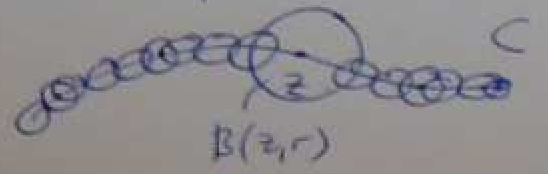
Lemma Let $f(z)$ be continuous on a domain G, C piecewise smooth curve in G. For any $\epsilon > 0$, there is a polygonal curve L determined by C such that

$$\left| \int_C f(z)dz - \int_L f(z)dz \right| < \epsilon$$

Proof (sketch) **claim** there is a bounded domain D s.t. $C \subseteq D \subseteq G$ s.t. $\overline{D} \subseteq G$.



proof (of claim) for each point $z \in C$ there is a neighborhood $B(z,r) = \{w \mid |w-z| < r\}$, $r > 0$ s.t. $z \in B(z,r) \subseteq G$



• wma $r < 1$ we may assume wma

without loss of generality wlog

$\bigcup_{z \in C} B(z, \frac{r}{2}) = D \leftarrow$ domain bounded diam(D) $\leq$ diam$(C) + 2$.

• $\overline{D} \subseteq \bigcup \overline{B(z, \frac{r}{2})} \subseteq G$ so $\overline{D} \subseteq G$.

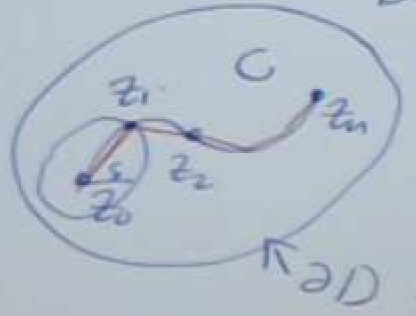
• let $\rho =$ distance from C to $\partial \overline{D}$ <u>claim</u> $\rho > 0$ (hw Q for next week)

• $f(z)$ is cts on G, so $f(z)$ cts on $\overline{D} \leftarrow$ closed and bounded

$\Rightarrow f(z)$ is <u>uniformly continuous</u> on $\overline{D}$

<u>recall</u> for any $\epsilon > 0$, there is a $\delta > 0$ s.t. for all $z_1, z_2 \in \bar{D}$, $|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \frac{\epsilon}{2\ell}$

ℓ = length of C (wlog $\delta < \rho$).



now partition C by $z_0, z_1, \ldots, z_n$

s.t. $|z_i - z_{i-1}| < \delta$

<u>note</u> the straight line segments connecting z_i to z_{i+1} all lie in D. ⊕ equal as integrating constant is path independent. $C = \cup \gamma_i$, $L = \cup \sigma_i$

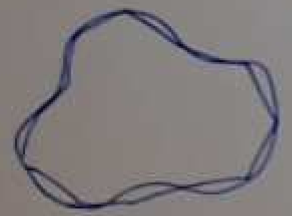
approx C $\left| \int_C f(z)dz - \sum_i \int_{\gamma_i} f(z_i)dz \right| \leq \sum_i \int_{\gamma_i} |f(z) - f(z_i)| dz \leq \ell \frac{\epsilon}{2\ell} = \frac{\epsilon}{2}$

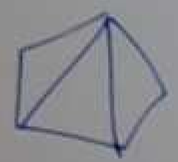
$\updownarrow$ ⊕

approx L $\left| \int_L f(z)dz - \sum_i \int_{\sigma_i} f(z_i)dz \right| \leq \sum_i \int_{\sigma_i} |f(z) - f(z_i)| dz \leq \ell \frac{\epsilon}{2\ell} = \frac{\epsilon}{2}$

so by triangle inequality $\left| \int_C f(z)dz - \int_L f(z)dz \right| \leq \epsilon$.

can approximate a piecewise smooth curve by straight line segments.





for closed curves can approximate these by triangles.

<u>Fact</u> Let $f(z)$ be continuous in a domain G containing a polygonal closed curve L. Then there are triangles $\Delta_1, \Delta_2, \ldots \Delta_n$ s.t.

$$\int_L f(z)dz = \int_{\Delta_1} f(z)dz + \int_{\Delta_2} f(z)dz + \cdots + \int_{\Delta_n} f(z)dz.$$