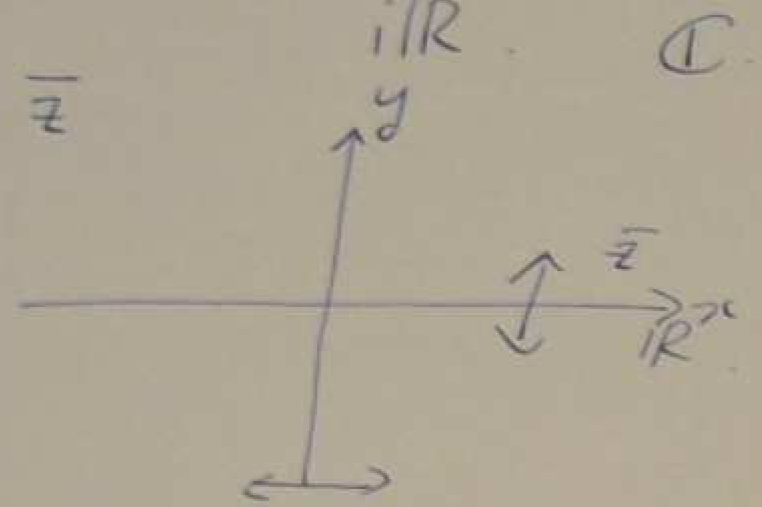
$\textcircled{1}$

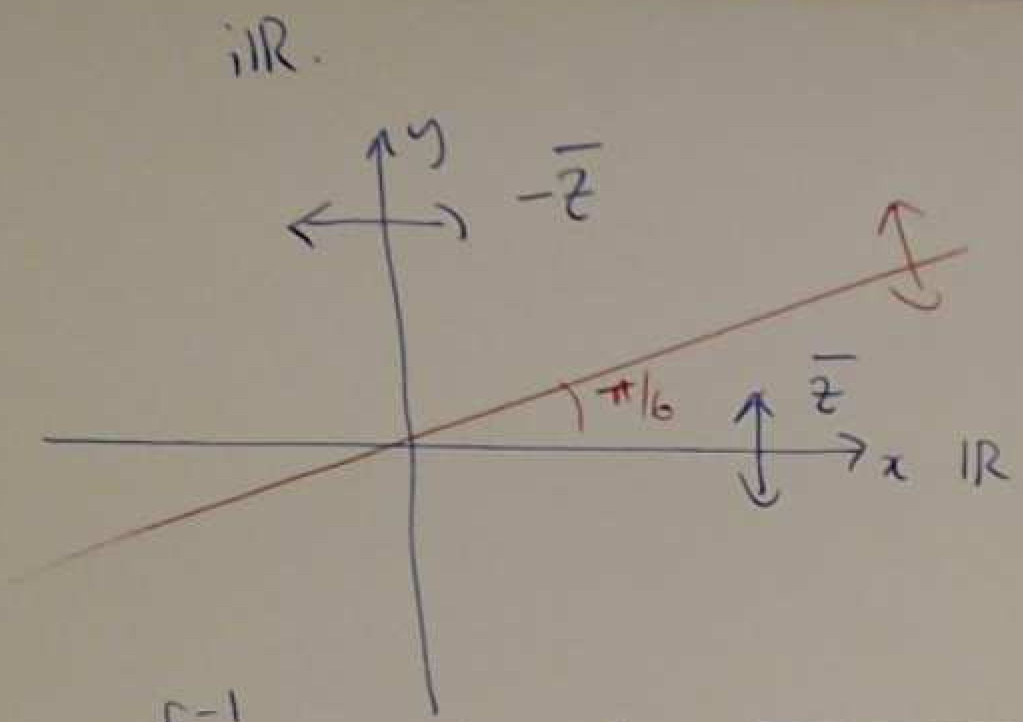
- do the HW.
- meet me on zoom to collect it.

<u>HW 1</u> (1) $re^{i\theta}$ ← useful.

(2) fgf^{-1} <u>example</u> $z \longmapsto \bar{z}$



$$z \longmapsto \bar{z}$$
$$x+iy \longmapsto x-iy$$



$f g f^{-1}$

$g: z \mapsto \bar{z}$

~~$f: z \mapsto e^{i\frac{\pi}{2}}$~~

$f: z \mapsto e^{i\frac{\pi}{2}} z$

$f^{-1}: z \mapsto e^{-i\frac{\pi}{2}} z$

$$z \overset{f^{-1}}{\longmapsto} e^{-i\frac{\pi}{2}} z \overset{g}{\longmapsto} \overline{e^{-\frac{\pi}{2} i} z}$$

$$= e^{\frac{\pi}{2} i} \bar{z} \overset{f}{\longmapsto} e^{i\frac{\pi}{2}} e^{i\frac{\pi}{2}} \bar{z}$$

$$e^{i\pi} \bar{z}$$

$$-\bar{z}$$

replace $\frac{\pi}{2}$ by $\frac{\pi}{6}$.

3

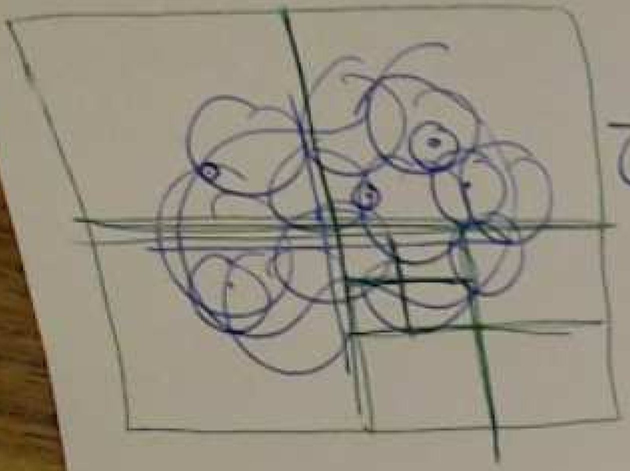
recall uniform continuity $f: G \to \mathbb{C}$

for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $z_1, z_2 \in G$, if $|z_1 - z_2| < \delta$ then $|f(z_1) - f(z_2)| < \epsilon$.

Thm (Heine-Borel) $\overline{G}$ bounded closed domain
$\overline{G} = G \cup \partial G$. G domain (open path-conn)

For each $z \overline{G}$ choose a disc $D_z = \{ w \mid |w - z| < r_z \}$.

Then there is a finite collection of discs $D_{z_1}, \dots, D_{z_n}$ which cover $\overline{G}$, i.e. $\overline{G} \subset \bigcup_{i=1}^{n} D_{z_i}$



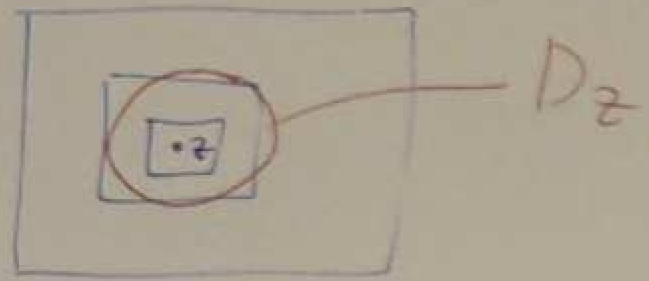
Proof $\bar G$ ~~[illegible]~~ bounded

$\Rightarrow \bar G \subseteq$ rectangle R_1

suppose $\bar G$ is not covered by finitely many discs. Divide R_1 into 4 congruent rectangles, at least one of these contains part of $\bar G$ which can only be covered by infinitely many discs, call this R_2, now continue, get a nested sequence $R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots$ s.t.

$R_i \cap \bar G$ can only be covered by infinitely many discs.

recall $\bigcap_{i=1}^{\infty} R_i = \{z\}$. consider the disc D_z has positive radius.



eventually $R_k \subset D_z$ for k large enough.

(diam $R_k < \frac{1}{2}$ radius of D_z).

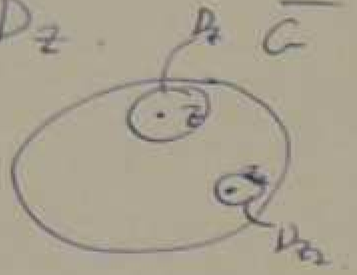
$\#$ $\overline{G} \cap R_k$ is containing in one disc D_z $\#$ $\square$

Thm If f is continuous on a closed bounded domain $\bar{G}$, then f is ~~absolutely~~ uniformly continuous on $\bar{G}$.

Proof suppose f is cts on $\bar{G}$ pick $z \in \bar{G}$

for any $\frac{\epsilon}{2} > 0$ there is a disc $D_z = |z' - z| < \delta_z$

s.t. $|f(z') - f(z)| < \frac{\epsilon}{2}$ for all $z' \in D_z$.

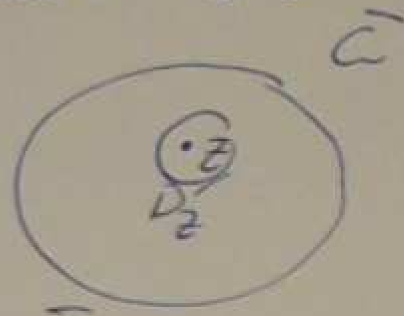


- let z' and $z'' \in D_z$

$|f(z') - f(z'')| = |f(z') - f(z) + (f(z) - f(z''))|$

$\le \underbrace{|f(z') - f(z)|}_{\epsilon/2} + \underbrace{|f(z) - f(z'')|}_{\epsilon/2} \le \epsilon$

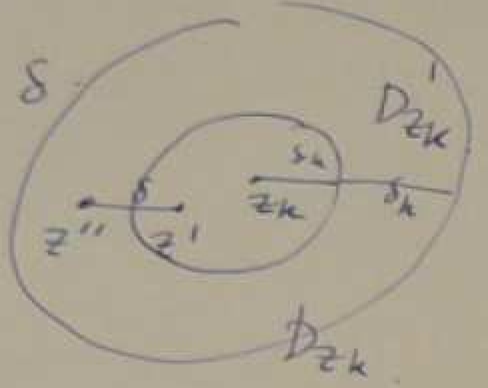
now replace D_z by D_z' ← disc of $\frac{1}{2}$ radius $= \frac{1}{2}\delta_z$



By [HB] there is a finite collection of discs $D'_{z_1}, \dots, D'_{z_n}$ which cover $\bar{C}$.

choose $\delta =$ max radius of any $D'_{z_1}, \dots, D'_{z_n}$

for any z', z'' with $|z'-z''| < \delta$



$z' \in D'_{z_k}$ for some k.

$|z'-z''| < \delta$ both $z', z'' \in D_{z_k}$.

$\Rightarrow |f(z')-f(z'')| \leq \epsilon$ $\square$

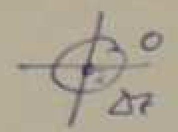
§4 Differentiation

$f : G \to \mathbb{C}$

Defn: f defined on a domain $G \subseteq \mathbb{C}$ is **differentiable** at $z \in G$ if the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists and is finite



call this the **derivative** $f'(z)$

Warning: this is totally different from $f : \mathbb{R}^2 \to \mathbb{R}^2$ being differentiable complex differentiable.

Defn we say f is analytic on G if it is differentiable at every point $z \in G$.
and analytic at z if it is differentiable in a neighbourhood of z.

Example $f : \mathbb{C} \longrightarrow \mathbb{C}$ $\quad z \longmapsto z^2$

this is differentiable

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

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10

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z = 2z.$$

$$f(z) = z^2$$
$$f'(z) = 2z.$$

Non-example

$$f : \mathbb{C} \longrightarrow \mathbb{C}.$$
$$z \longmapsto \mathrm{Re}(z)$$
$$x+iy \longmapsto x.$$

can check: this is continuous.

claim: not (complex) differentiable anywhere.

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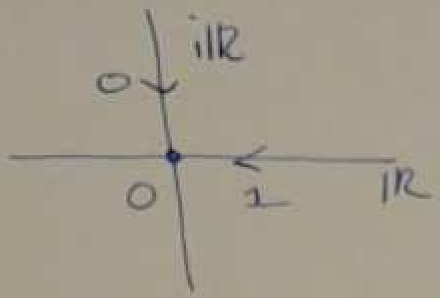
$z \mapsto \mathrm{Re}(z)$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\mathrm{Re}(z+\Delta z)-\mathrm{Re}(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\cancel{\mathrm{Re}(z)} + \mathrm{Re}(\Delta z) - \cancel{\mathrm{Re}(z)}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\mathrm{Re}(\Delta z)}{\Delta z}$$

DNE

$\mathbb{C}$.



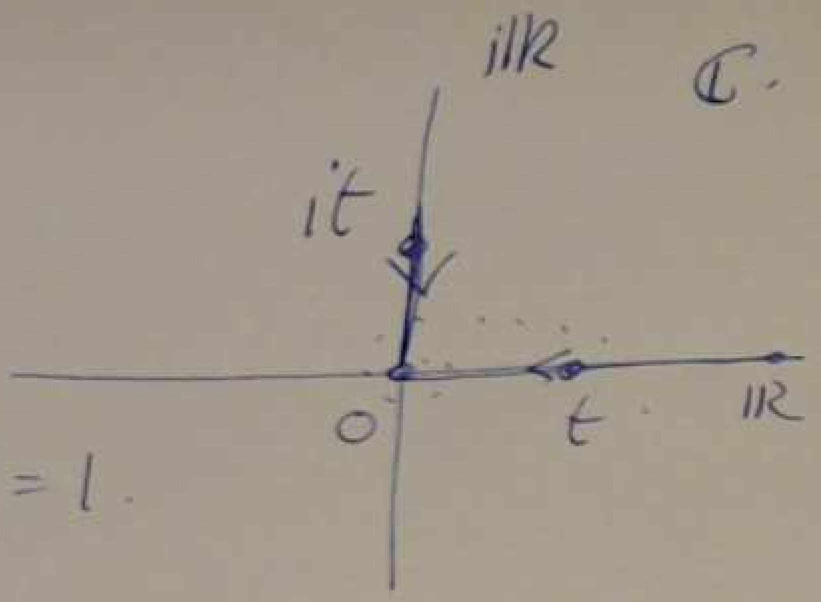
$\Delta z = t \in \mathbb{R}$.

$$\lim_{t \to 0} \frac{t}{t} = 1$$

$\Delta z = it \in i\mathbb{R}$

$$\lim_{t \to 0} \frac{0}{t} = 0$$

$$\lim_{\Delta z \to 0} \frac{\mathrm{Re}(\Delta z)}{\Delta z}$$



$t \in \mathbb{R}$. $\quad t \to 0 \quad \frac{\mathrm{Re}(t)}{t} = \frac{t}{t} = 1$.

$it \quad t \in \mathbb{R} \quad \frac{\mathrm{Re}(it)}{t} = \frac{0}{t} = 0$

$z \mapsto \mathrm{Re}(z) \leftarrow$ not complex differentiable anywhere!

$x+iy \mapsto x$.

<u>note</u>

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (u(x,y), v(x,y))$$

Df = matrix of partial derivatives.

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

the derivative is a linear map.

$$Df: \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$f: \mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longmapsto \lambda z$$

$\lambda \in \mathbb{C}$. $\lambda = \lambda_1 + \lambda_2 i$

$\mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto \lambda z$

$x+iy \mapsto (a+bi)(x+iy)$

$= ax - by + i(bx+ay)$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \underbrace{\begin{bmatrix} a & -b \\ b & a \end{bmatrix}}_{\uparrow\, 2d} \begin{bmatrix} x \\ y \end{bmatrix}$$

general linear map

$\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\uparrow\, 4d} \begin{bmatrix} x \\ y \end{bmatrix}$$

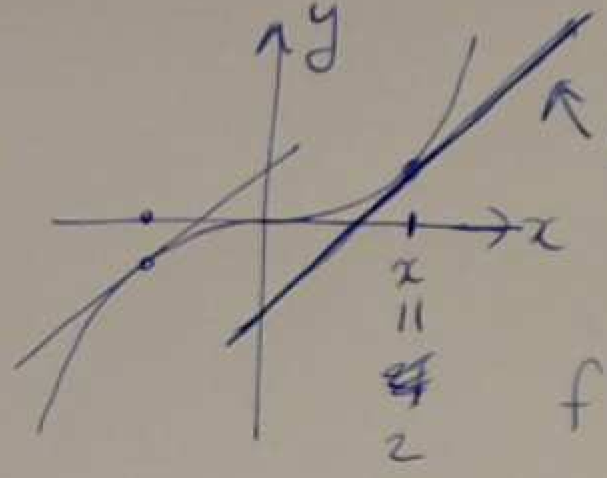
$f : \mathbb{C} \longrightarrow \mathbb{C}$ $\Rightarrow$ $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

complex diff

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto x^3.$

$f'(x) = 3x^2$



slope of tangent line

$f'(x) = 3x^2.$

$f'(4) = 12.$

$\mathbb{R} \to \mathbb{R}$

$t \mapsto f'(x)\,t$

$t \mapsto 3x^2 t.$

$\mathbb{R} \to \mathbb{R}$

$t \mapsto 12t.$

$f : \mathbb{R}^2 \to \mathbb{R}^2.$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}.$

$Df : \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

<u>Exercise</u> show $f : z \mapsto \overline{z}$ <u>not</u> complex differentiable.

Standard derivative rules hold.

$$(cf(z))' = cf'(z) \quad c \text{ const}$$

$$(f(z)+g(z))' = f'(z) + g'(z)$$

$$(f(z)\,g(z))' = f'(z)\,g(z) + f(z)\,g'(z)$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{g'(z)\,f(z) - g(z)\,f'(z)}{g(z)^2}.$$

16

<u>Exercise</u> show $f: z \mapsto \bar{z}$ <u>not</u> complex differentiable.

Standard derivative rules hold.

$$(cf(z))' = cf'(z) \quad c \text{ const}$$

$$(f(z)+g(z))' = f'(z) + g'(z)$$

$$(f(z)\,g(z))' = f'(z)\,g(z) + f(z)\,g'(z)$$

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{g(z)\,f'(z) - g'(z)\,f(z)}{g(z)^2}.$$

17

$$(f(g(z)))' = f'(g(z))\,g'(z).$$

$$(z^n)' = n z^{n-1}$$

$\Rightarrow$ all polynomials are analytic

$\Rightarrow$ all rational functions are analytic

$\dfrac{p(z)}{q(z)}$ (where they are defined)

$(q(z) \neq 0)$

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Example $\quad f(z) = z^2 + z + 1$

$$f'(z) = 2z + 1$$

fact $\quad e^z$ is complex differentiable/ analytic

$$(e^z)' = e^z \qquad 1, 2, 4, 8, 16, \dots$$

$$f(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$f'(z) = \frac{2z}{2!} + \frac{4z^3}{4!} + \cdots$$

Example

$$f(z) = z^2 + z + 1$$

$$f'(z) = 2z + 1$$

fact e^z is complex differentiable/ analytic

$$(e^z)' = e^z \qquad 1, 2, 4, 8, 16, \dots$$

$$f(z) = \cancel{1 + \frac{z^2}{(2)!} +} \; 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$f'(z) = \frac{2z}{2!} + \frac{4z^3}{4!} + \cdots$$

Complex differentials

$f : \mathbb{C} \to \mathbb{C}$ differentiable

then derivative $f'(z)$ is ~~locally~~ for fixed z is a linear map

$$\mathbb{C} \longrightarrow \mathbb{C}$$
$$\Delta z \mapsto f'(z)\Delta z.$$

text: $w = f(z)$

Δz for the local variable

$$df = dw = f'(z)\,\Delta z \quad (*)$$

special case $f(z) = z$

$$dz = (z')\Delta z = \Delta z.$$

so we can write dz for Δz, $(*)$ $df = f'(z)\,dz$

Cauchy-Riemann equations

consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad Df = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}.$$

$$f : \mathbb{C} \longrightarrow \mathbb{C}.$$

$$z \longmapsto f(z)$$

$$x+iy \mapsto u(x,y) + iv(x,y).$$

$\underline{\text{Thm}}$ [CR] $f(z)$ is (complex) differentiable at $z = x+iy$ iff the real valued functions u, v are differentiable at (x,y) $\underline{\underline{\text{and}}}$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy Riemann equations

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~~Proof~~ intuition

linear maps.

$M : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}$

$M : \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto \lambda z$ $\quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$(x+iy) \longmapsto (a+bi)(x+iy)$

$ax - by + i(ay + bx)$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{also} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

$\square$

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<u>note.</u> $\mathbb{R}^2 \subseteq \mathbb{R}^4$

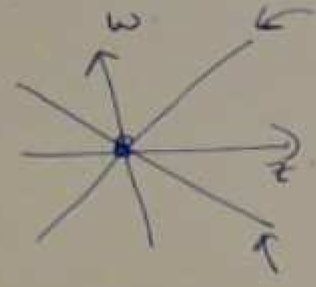
$\mathbb{C} \subseteq \mathbb{C}^2$ totally different.

$\mathbb{R}^4$ (x, y, z, t) $\leftarrow$ plane $(x, y, 0, 0)$ $\cap$ 1-d line

plane $(0, y, z, 0)$

$(0, y, 0, 0)$

$\mathbb{C}^2$



$\leftarrow$ $w = \lambda z$

$w = \mu z$

intersection $\lambda z = \mu z$

$(\lambda - \mu) z = 0$

$\neq 0$

$\Rightarrow z = 0$

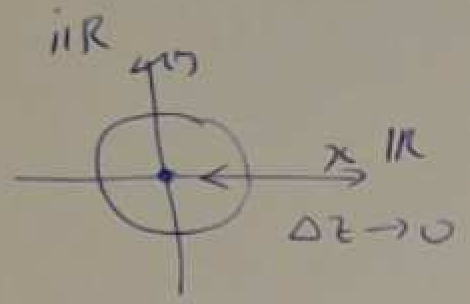
Fact $f : \mathbb{C} \longrightarrow \mathbb{C}$

$$z \longmapsto f(z) = \cancel{u(x,y)} \quad u(x+iy) + iv(x+iy)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

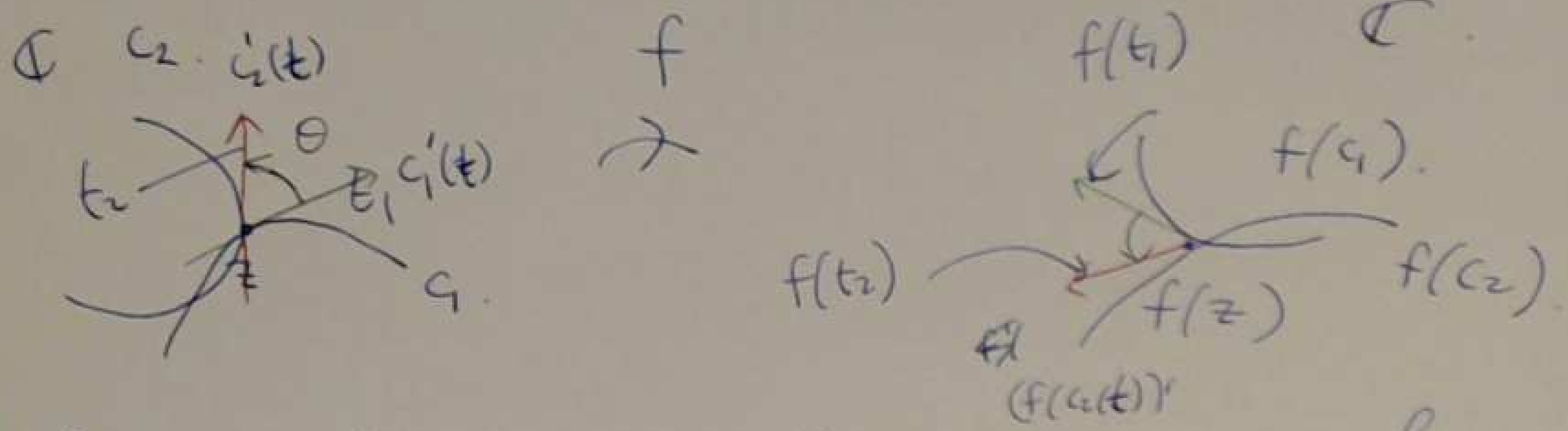
$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$



conformal maps

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto f(z).$$

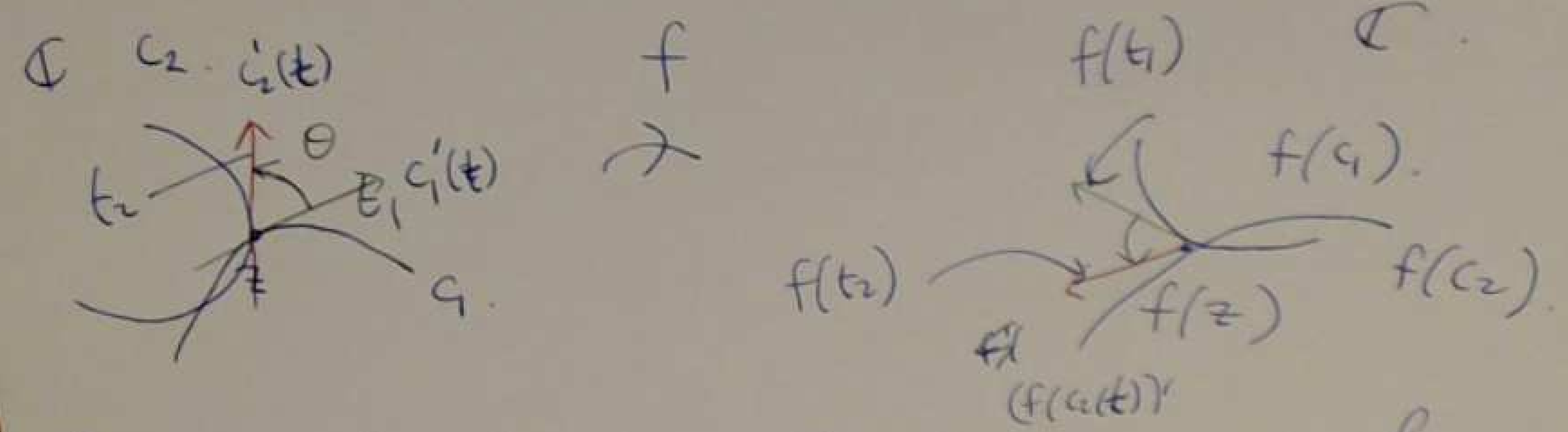


f is conformal at z if for every pair of tangent vectors t_1, t_2, the angle between t_1 and t_2 has the same size and direction as the angle between $f(t_1)$ and $f(t_2)$

(24)

conformal maps

$$f : \mathbb{C} \longrightarrow \mathbb{C}.$$
$$z \longmapsto f(z).$$



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