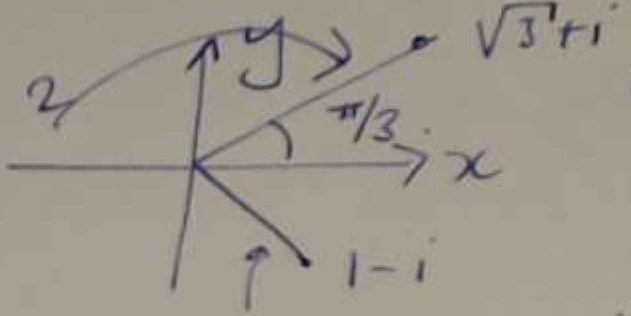
Q20 i) $\sqrt[6]{\dfrac{1-i}{\sqrt{3}+i}}$

$$\frac{1-i}{\sqrt{3}+i}$$



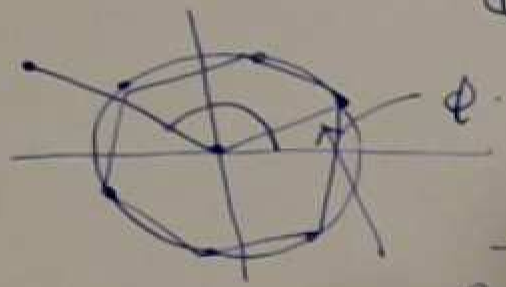
$\mathbb{C}$

$\sqrt{2}e^{-i\pi/4}$

$2e^{i\pi/3}$

$$\frac{\sqrt{2}\,e^{-\pi/4\, i}}{2\,e^{i\pi/3}} = \frac{\sqrt{2}}{2}e^{-\frac{7\pi}{12}i} = \frac{\sqrt{2}}{2}e^{\frac{5\pi}{12}i}$$

$\mathbb{C}$



$$2^{-1/2}e^{\frac{5\pi}{12}i}$$

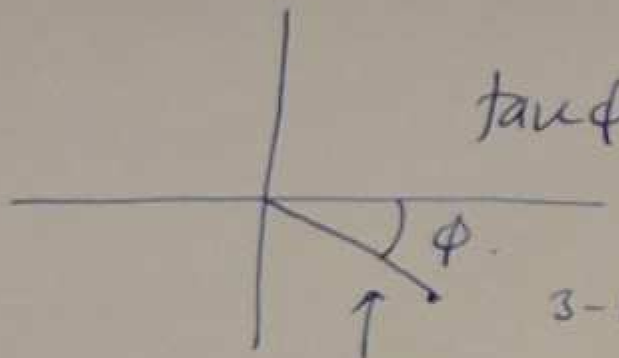
$$2^{-1/12}e^{\frac{5\pi}{72}i}$$

$$\phi + \frac{2\pi k}{6}$$

Q15 c) $\left(1 - \frac{\sqrt{3-i}}{2}\right)^{24}$.

$\mathbb{C}$.

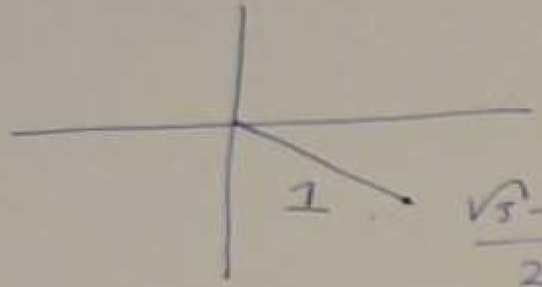
$\tan\phi = \frac{1}{3}$.



$\left(1 - \frac{\sqrt{3}-i}{2}\right)^{24}$.

$\left(1 - \frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{24}$

$\left(\frac{2-\sqrt{3}}{2} + \frac{i}{2}\right)^{24}$ $\quad re^{i\theta}$.

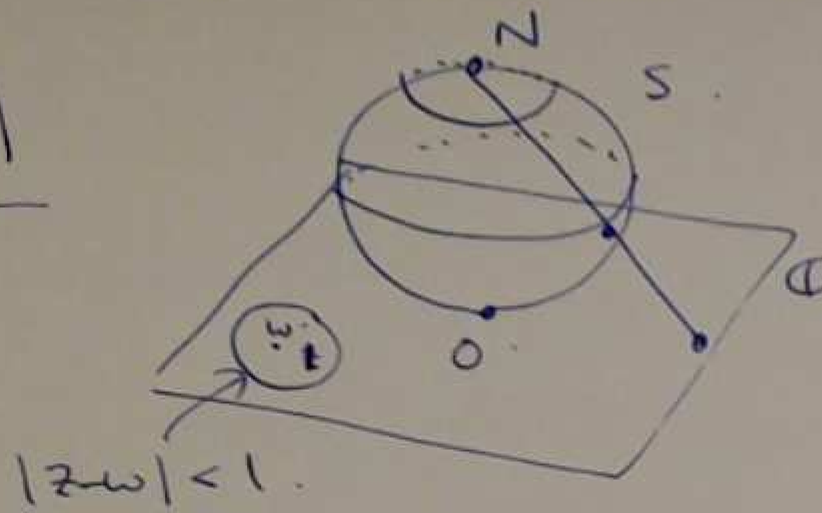


$e^{-\pi/3}$.

$\frac{1}{4}\left((2-\sqrt{3})^2+1\right)$

$(2-\sqrt{3})^{12}$.

recall



$S \leftrightarrow \mathbb{C} \cup \{\infty\}$.

neighbourhoods of N are $\{z \mid |z| > M\} = D_M$

∞ is a limit point of a sequence (z_n) if any D_M contains infinitely many points of (z_n).

§3 Complex functions

recall $f: A \to B$ is a function if the set of pairs $(a, f(a)) \subseteq A \times B$ satisfies:

for every $a \in A$ there is exactly one pair $(a, f(a))$.

notation: $a \mapsto f(a)$ $\qquad f: A \longrightarrow B$

domain (for A), range (codomain) (for B)

for us $f: \mathbb{C} \longrightarrow \mathbb{C}$, $\quad \mathbb{C} \supseteq G$

$\mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C} \cup \{\infty\}$

Remark $\mathbb{C} \cong \mathbb{R}^2$ so $f: \mathbb{C} \longrightarrow \mathbb{C}$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Recall $f: \mathbb{C} \longrightarrow \mathbb{C}$ is one-to-one / injective if

$$f(a) = f(b) \Rightarrow a = b.$$

if f is injective then there is an inverse function $f^{-1}: \mathbb{C} \longrightarrow \mathbb{C}$

in general f is not injective no inverse function

but pre-image $f^{-1}(z) = \{ w \in \mathbb{C} \mid f(w) = z \}$ exists.

Ex

$f: \mathbb{C} \to \mathbb{C}$ two-to-one except at $0 \mapsto 0$

$z \mapsto z^2$ $\infty \mapsto \infty$

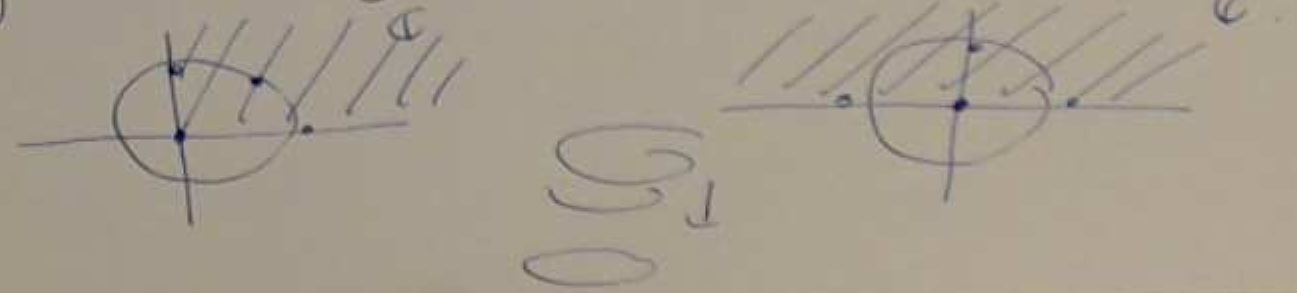
$f^{-1}(z) = \{$ two square roots of $z \}$.

$\mathbb{R}^2 \to \mathbb{R}^2$

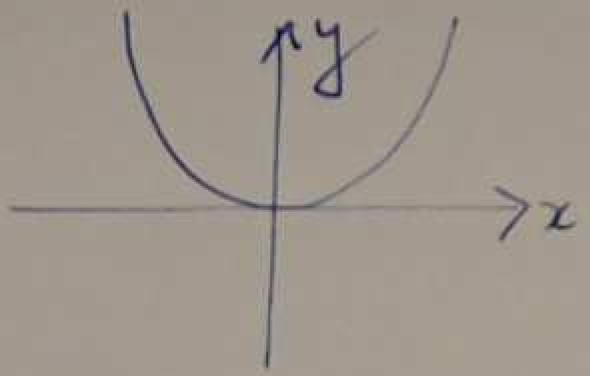
$x+iy \mapsto (x+iy)^2 = x^2 - y^2 + 2xyi$

Q: what does this look like geometrically?

— angle doubling map $\mathbb{C} \longrightarrow \mathbb{C}$.



$f: \mathbb{R} \longrightarrow \mathbb{R}$.
$x \longmapsto x^2$



$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$f: \mathbb{C} \longrightarrow \mathbb{C}$
$z \longmapsto z^2$

graph lives in $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2 \cong \mathbb{R}^4$.

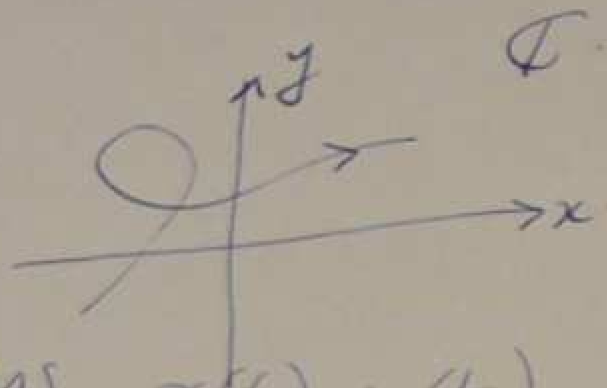
harder to visualize in 4d.

Curves

a parameterized curve is a map

$$I=[a,b] \xrightarrow{f} \mathbb{C}.$$

$\subseteq \mathbb{R}.$



- any two two continuous functions $x(t), y(t)$ determine a curve.

$$t \longmapsto (x(t), y(t))$$

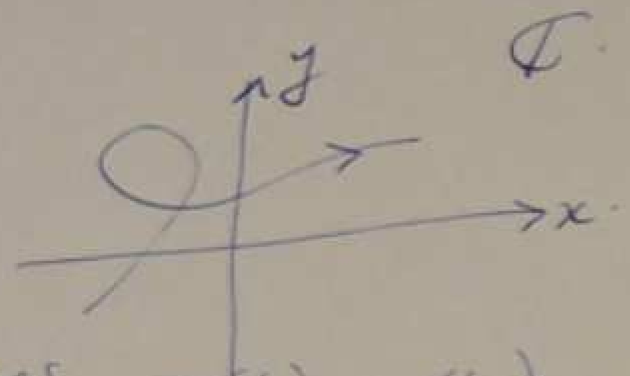
$$t \longmapsto z(t) = x(t) + iy(t)$$

- the curve is closed (loop) if the first point and the last point coincide, i.e. $f(a) = f(b)$


$f(I)$

Curves a parameterized curve is a map $I = [a,b] \xrightarrow{f} \mathbb{C}$.

$\subseteq \mathbb{R}$.



- any two two continuous functions $x(t), y(t)$ determine a curve.

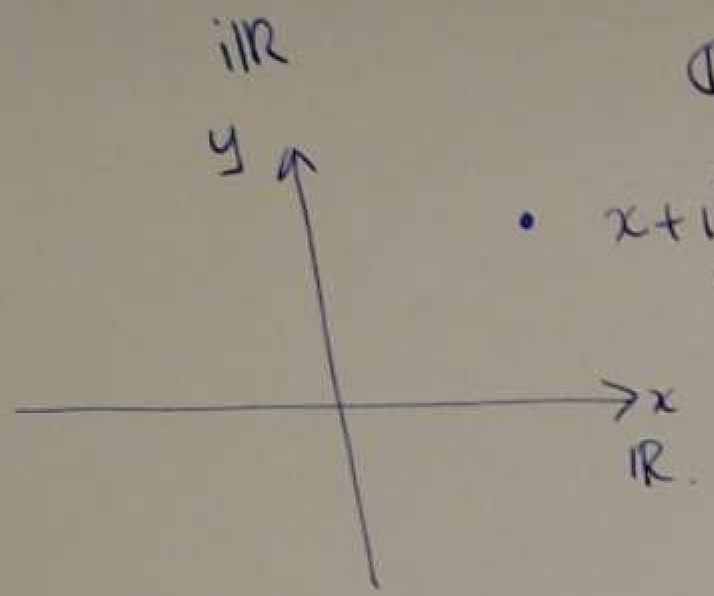
$$t \longmapsto (x(t), y(t))$$

$$t \longmapsto z(t) = x(t) + iy(t)$$

- the curve is closed (loop) if the first point and the last point coincide, i.e. $f(a) = f(b)$

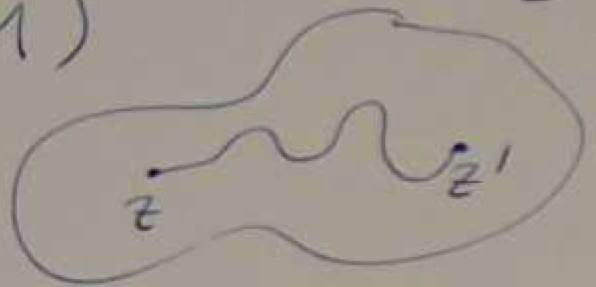
$f(I)$ otherwise, call it an arc or a path

$\mathbb{C}$. $\mathbb{R}^2$



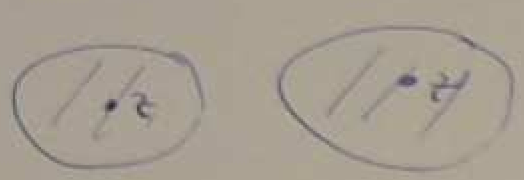
• $x+iy$ (x,y)

$E \subseteq \mathbb{C}$.



Let $E \subseteq \mathbb{C}$ E is arc/path-connected if any pair of points z, z' in E can be connected by a path in E.

<u>not</u> path connected

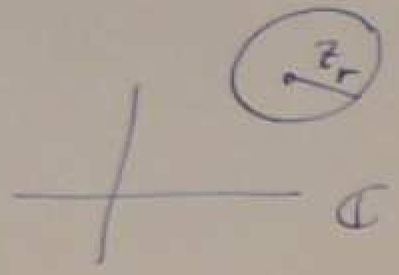


Also path connected

Recall a neighbourhood of

$z \in \mathbb{C}$ is an open disc

centered at z, i.e. $\{w \in \mathbb{C} \mid |z-w| < r\}$

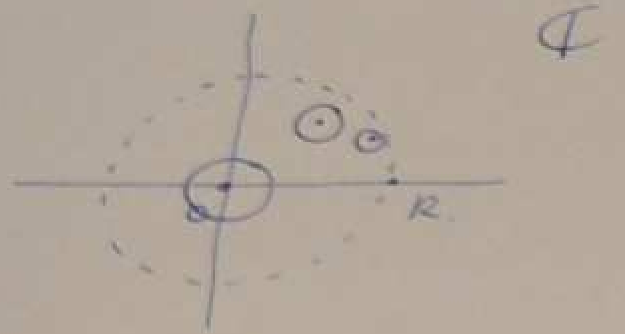


Def^n A set $E \subset \mathbb{C}$ is open if

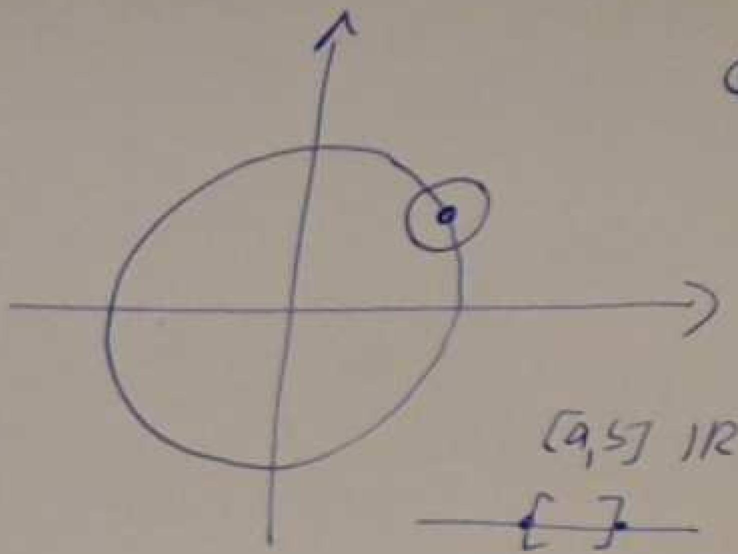
every point $z \in E$ contains a

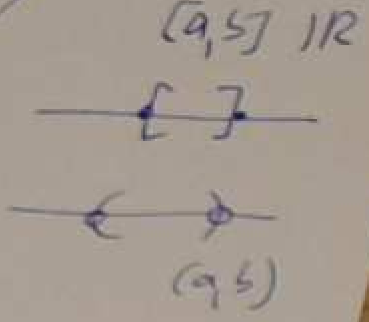
neighbourhood in E.

Example open disc $|z| < R$



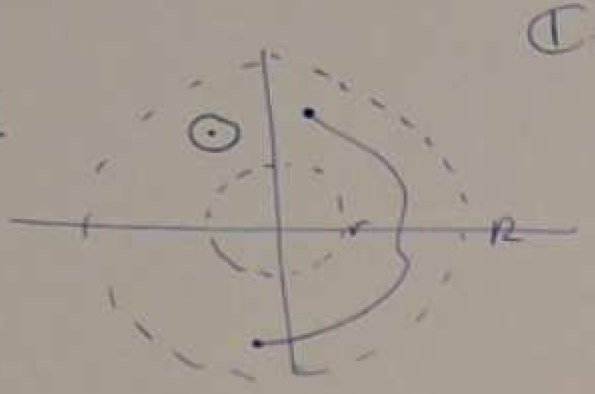
not open: $\{z \mid |z| \le R\}$.





Def^n A set $G \subseteq \mathbb{C}$ is a (complex) domain if it is path-connected and open.

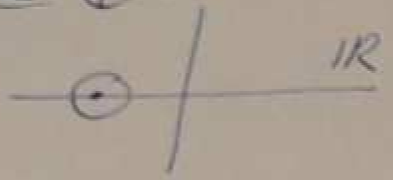
Example



$r < |z| < R$

Non-example

$\mathbb{R} \subseteq \mathbb{C}$.



Notation set complements

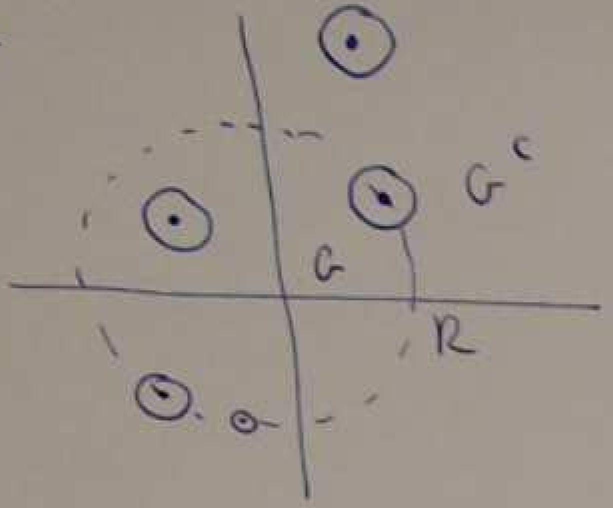
$G \subseteq \mathbb{C}$.

G^c complement $= \mathbb{C} \setminus G$.

Defn Let $z \in G^c$ If there is a neighbarhood of z contained in the complement, then z is an *exterior* point of G. otherwise, if every neighbarhood of z contains points of both G and G^c, z is a boundary point of G.

(13)

Ex



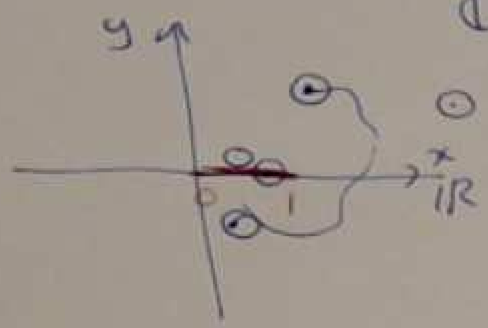
$$G = \{ z \mid |z| < R \}$$

$$G^c = \{ z \mid |z| \geq R \}$$

boundary of G notation ∂G

$$|z| = R.$$

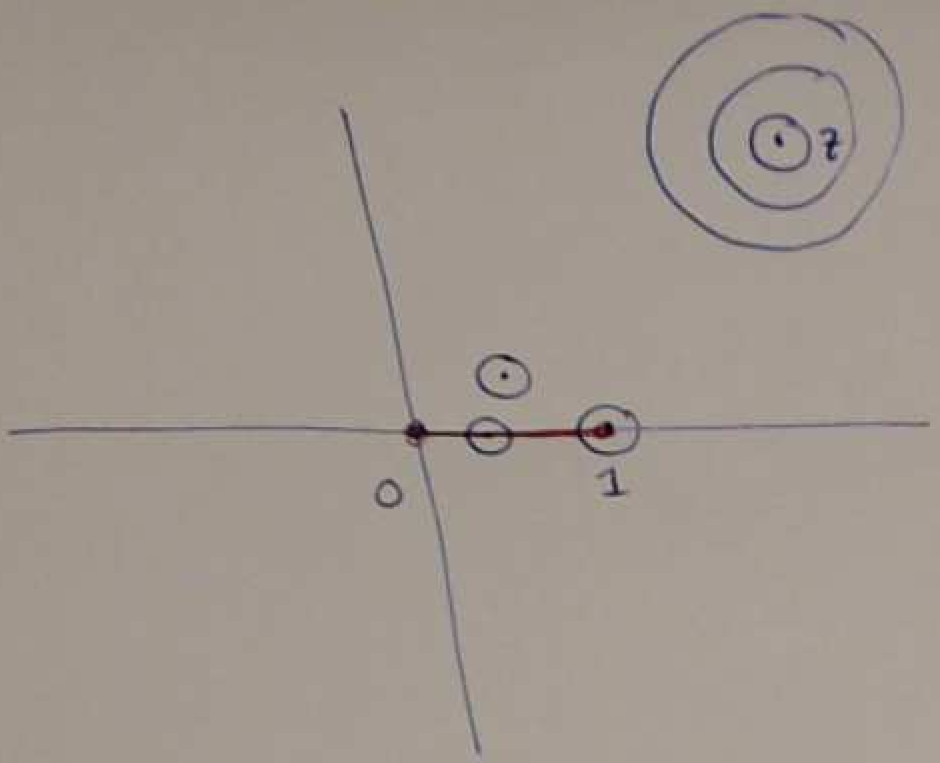
Ex. $G = \mathbb{C} \setminus [0,1]$ ← domain. open, path connected



$$\partial G = [0,1] \subseteq \mathbb{R}.$$

exterior$(G) = \phi$ ← empty set

$\mathbb{C}$



G

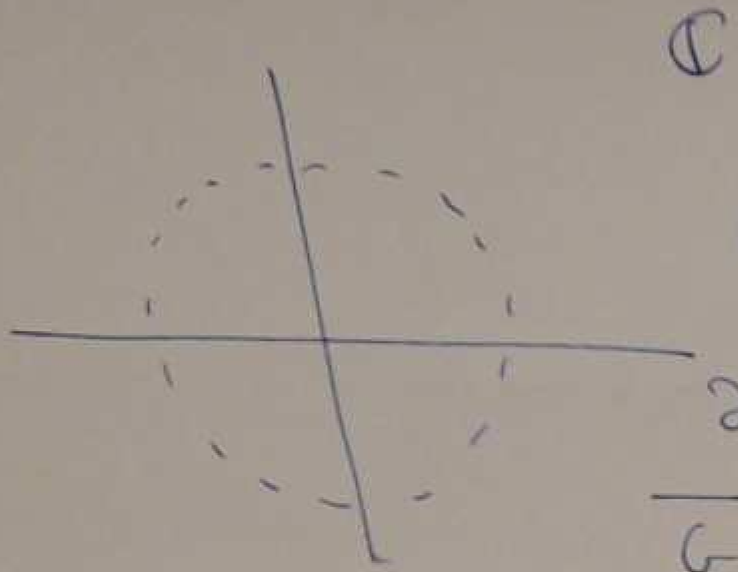
G^c

$\partial G = [0,1]$

<u>Defn</u> given a domain (open path connected) G, let $\overline{G} = G \cup \partial G$

this is called a <u>closed</u> domain.

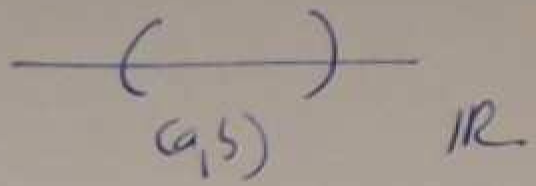
15

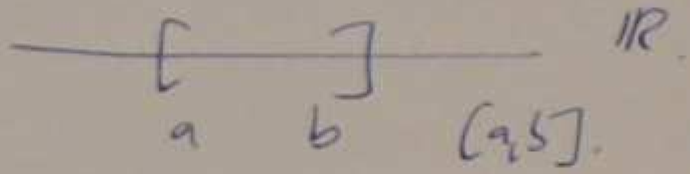


$G = |z| < R$

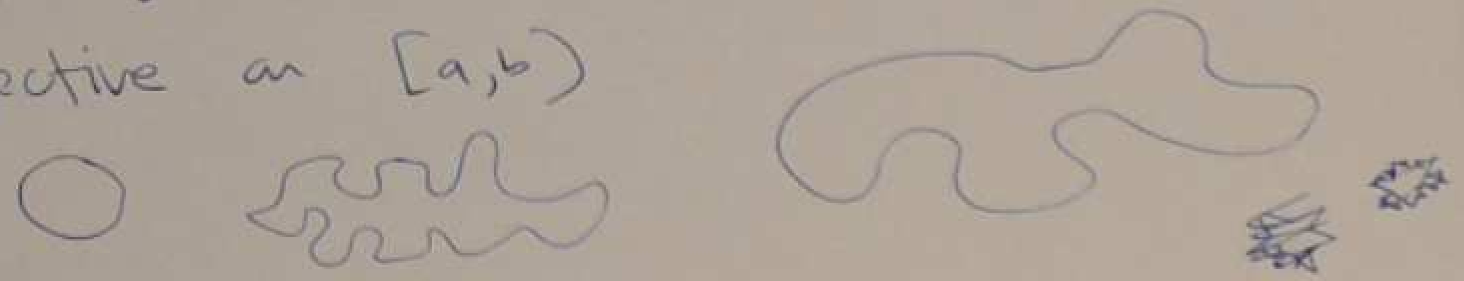
$\partial G = |z| = R$

$\overline{G} = |z| \leq R$





Def$^{\underline{n}}$ Let $f : \underset{[a,b]}{I} \longrightarrow \mathbb{C}$ be a closed curve i.e. $f(a) = f(b)$. It is a Jordan curve if f is injective on $[a,b)$.



Thm [Jordan curve theorem] A Jordan curve C cuts the plane into two open domains, one bounded, called the interior, other one called the exterior. ext

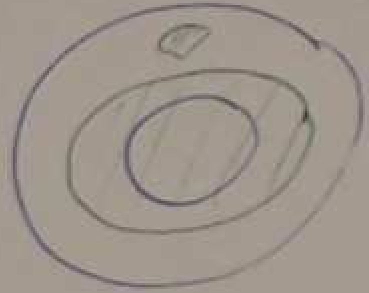
Defn A domain G is simply connected if every Jordan curve $C \subseteq G$ has its interior disc also contained in G.

otherwise not simply connected
(multiply-)

Examples.

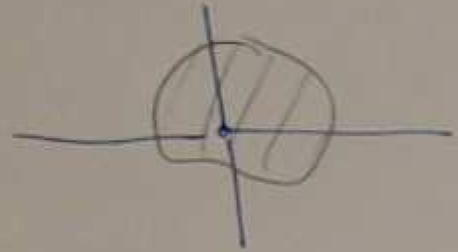
simply connected

not simply connected.



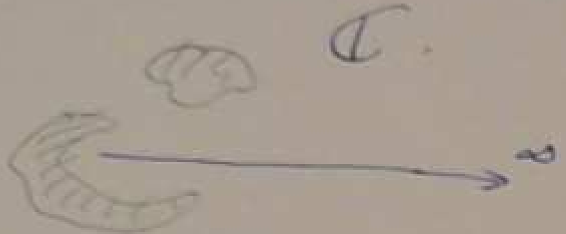
$r < |z| < R$

not simply connected $\mathbb{C} \setminus \{0\}$.



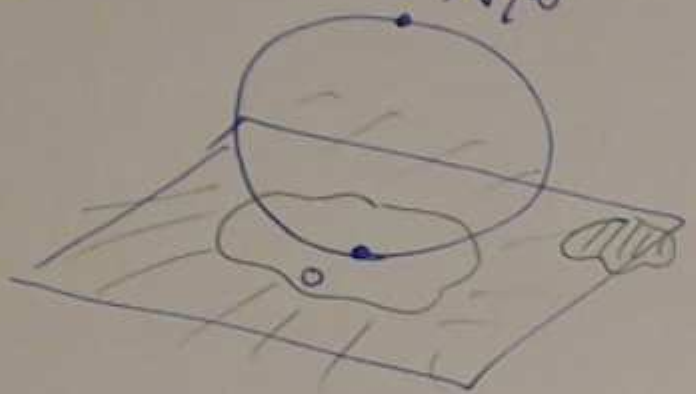
simply connected

$\mathbb{C} \setminus [0, \infty)$



Defn for $\mathbb{C} \cup \{\infty\}$: $G \subseteq \mathbb{C} \cup \{\infty\}$ is simply connected if any Jordan curve C in G, either interior$(C) \underset{\infty/0}{\subset} G$ or exterior of $C \subset G$.

Example

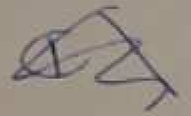


$\mathbb{C} \setminus \{0\}$.

$(\mathbb{C} \cup \{\infty\}) \setminus \{0\}$

simply connected.

Not simply connected:



$\mathbb{C} \cup \{\infty\} \setminus \{0, 1\}$

$\mathbb{C} \cup \{\infty\}$

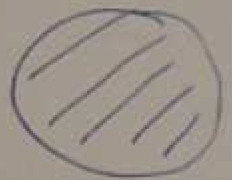
Zoom UI: Unpin Video | Recording...

Def$^{\underline{n}}$ Let C_0 be a Jordan curve, containing (19) $C_1, \dots, C_n$ Jordan curves with disjoint interiors.

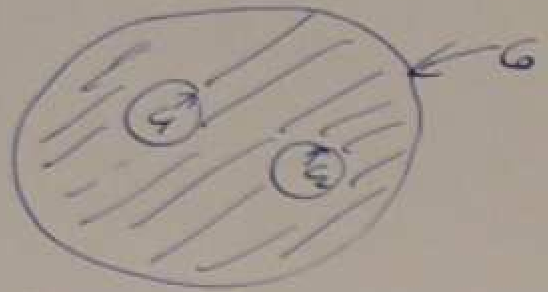
then ~~int(C) \ {C_1~~ $\operatorname{int}(C_0) \setminus \{\operatorname{int}(C_1), \dots, \operatorname{int}(C_n)\}$

this is $(n+1)$-connected

doubly connected



one-connected. simply connected.



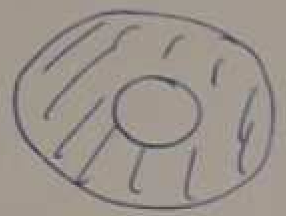
3-connected
triply connected

Joseph Maher

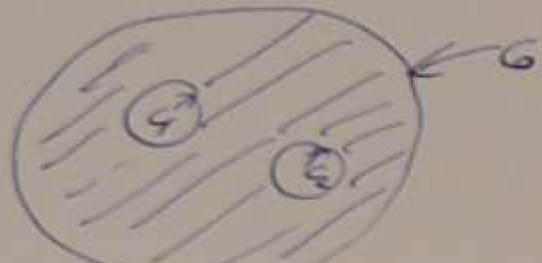
Def^n Let C_0 be a Jordan curve, containing $C_1, \dots, C_n$ Jordan curves with disjoint interiors

then ~~int(C) \ {C~~ $\text{int}(C_0) \setminus \{\text{int}(C_1), \dots, \text{int}(C_n)\}$

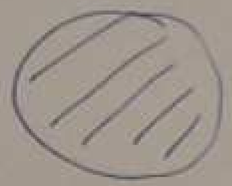
this is $(n+1)$-connected.



doubly connected



3-connected
triply connected



one-connected.
simply connected.

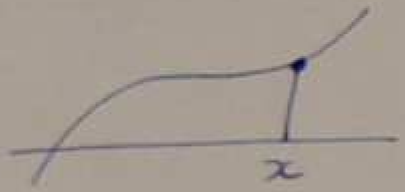
$f: \mathbb{R} \to \mathbb{R}$
$\frac{x+1}{x-1}$

$f: \mathbb{C} \to \mathbb{C}$
$z \mapsto \frac{z+1}{z-1}$

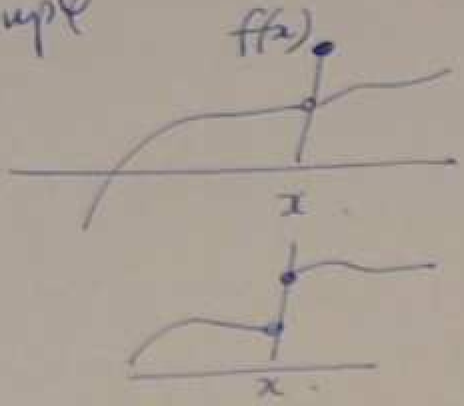
$f: A \to B$

Continuity Recall $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if $\lim_{y \to x} f(y) = f(x)$

Example



Nonexample



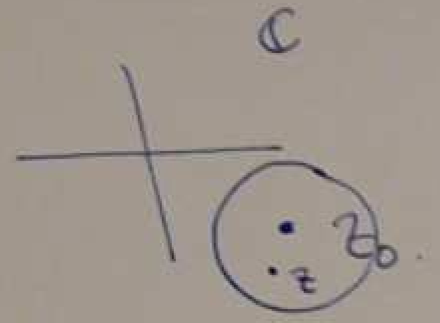
Defn $f : \underset{\text{domain}}{G} \to \mathbb{C}$

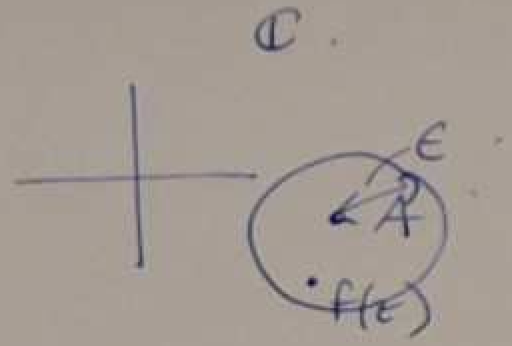
$\lim_{z \to z_0} f(z) = A$ notation $f(z) \to A$ as $z \to z_0$.

if for all $\epsilon > 0$, there is a $\delta > 0$

s.t. $|f(z) - A| < \epsilon$ for all $|z - z_0| < \delta$.

$f : \mathbb{C} \longrightarrow \mathbb{C}$.





$|z - z_0| < \delta$ $\qquad$ $|f(z) - A| < \epsilon$.

given $\epsilon > 0$, there is a δ

<u>Defn</u> Furthermore if $f(z_0) = \lim_{z \to z_0} f(z)$

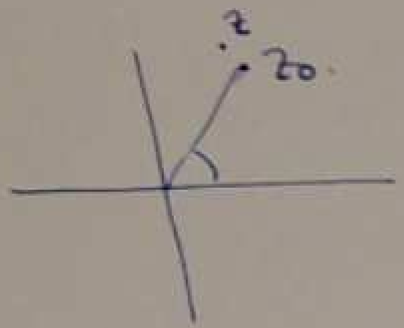
then f is continuous at z_0.

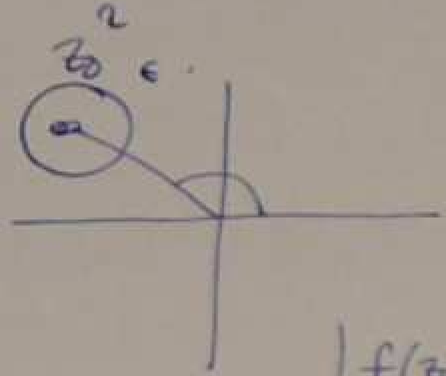
If f is continuous at all $z \in G$, then say f is continuous on G.

Example

$f : \mathbb{C} \longrightarrow \mathbb{C}$ show continuous.

$z \longmapsto z^2$





$|f(z_0) - f(z)| < \epsilon$

$$f(z) - f(z_0) = z^2 - z_0^2 = (z - z_0)(z + z_0)$$

$|ab| = |a||b|$

$$|f(z) - f(z_0)| \leq |z - z_0| \underbrace{|z + z_0|}_{\leq |z| + |z_0|}$$

$|z| -$

$$|z - z_0| < \delta \Rightarrow |z| < |z_0| + \delta$$

so $|f(z) - f(z_0)| < \underbrace{|z - z_0|}_{< \delta} (|z_0| + \delta)$

pick $\delta = \dfrac{\epsilon}{|z_0| + \delta}$

continuity for $f: \mathbb{C} \longrightarrow \mathbb{C}$.

for more general subsets, just restrict everything to $f: G \longrightarrow \mathbb{C}$.

$|z-z_0| < \delta$ or $|z-z_0| < \delta \cap G$.

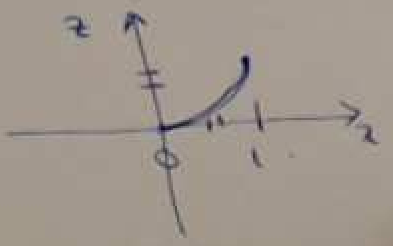
same definition, just restrict to points in G.

uniform continuity

$f(z)$ defined on $E \subseteq \mathbb{C}$ is uniformly continuous if for all $\epsilon > 0$, there is a $\delta > 0$ such that for all $z, z' \in E$, if $|z - z'| < \delta$ then

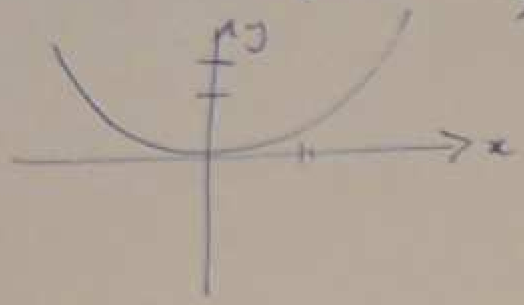
$$|f(z) - f(z')| < \epsilon.$$

Example $f : [0,1] \longrightarrow [0,1]$, $x \longmapsto x^2$



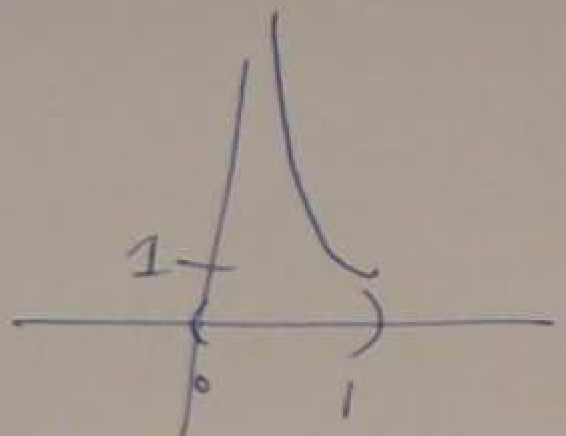
uniformly cts.

Non-example $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^2$



Non example: $f: \underset{\text{open}}{(0,1)} \longrightarrow \mathbb{R}$

$x \longmapsto \frac{1}{x}.$



not uniformly cts.

Fact f is continuous on closed bounded domain then f is uniformly continuous.