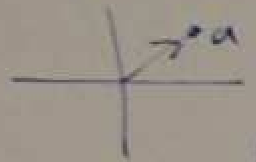
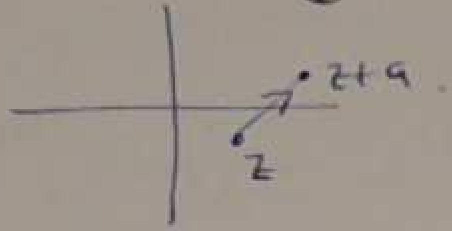
Q2 a) $z \mapsto z+a$

Q3a) f^{-1} fg fgf^{-1} $f = e^{i\theta}z$ $g = z+a$.

$\mathbb{C}$.

Q2 a) $z \mapsto z+a$





$z = x+iy$

$a = c+di$.

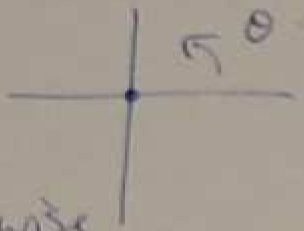
$z \mapsto z+a$.

translation by a. $(x,y) \mapsto (x+c, y+d)$

Q3 a) $f(z) = e^{i\theta}z$.

$z \mapsto e^{i\theta}z$.

rotation by θ anticlockwise



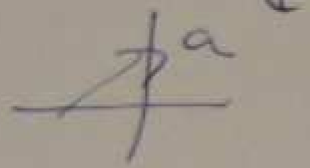
$z = se^{i\phi}$.

$e^{i\theta}z = e^{i\theta}se^{i\phi}$.

$= se^{i(\theta+\phi)}$

Q2a) $f(z) = e^{i\theta} z$ $\quad g(z) = z + a$. $\quad f^{-1}\, fg \quad fgf^{-1}$ $\quad$ 2

• $f^{-1}(z) = \dfrac{z}{e^{i\theta}} = e^{-i\theta} z$. ← clockwise rotation by θ



$$f(g(z)) = e^{i\theta}(z+a) = e^{i\theta}z + e^{i\theta}a.$$

$$fgf^{-1}(z) = e^{i\theta}\left(e^{-i\theta}z + a\right) = z + e^{i\theta}a.$$

translation by $\underbrace{e^{i\theta}a}_{f(a).}$

$g : w \mapsto g(w)$.

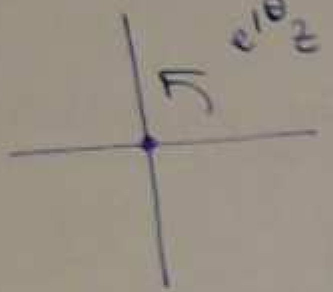
$fgf^{-1} : f(w) \mapsto fgf^{-1}f(w) = fg(w)$.

translating by the rotated vector $e^{i\theta}a$.

$z \mapsto e^{i\theta} z + e^{i\theta} a$.

$f(z) = e^{i\theta} z$.

$g(z) = z + a$.



$z = e^{i\theta} z \boxed{+ e^{i\theta} a}$

$z(1 - e^{i\theta}) = e^{i\theta} a$.

$z = \dfrac{e^{i\theta} a}{1 - e^{i\theta}}$.

.... this is a rotation about the fixed pt

$h(z) = e^{i\theta}$.

$t(z) = z + \dfrac{e^{i\theta} a}{1 - e^{i\theta}}$.

$t h t^{-1}$

$t^{-1} h t$

$f g \underline{f^{-1}(z)}$

$f g(e^{-i\theta} z)$

$f(e^{-i\theta} z + a)$

$e^{i\theta}(e^{-i\theta} z + a)$.

$z + e^{i\theta} a$.

$f(z) = e^{i\theta}$ $g(z) = z + a$.

$g f g^{-1}(z) = e^{i\theta}(z - a) + a$

$= e^{i\theta} z \boxed{+ a - a e^{i\theta}}$

rotation by θ around a.

fixed pt of f is 0

$f(0) = 0$.

fixed pt of $g f g^{-1}(g(0))$

$g f g^{-1} g(0) = g f(0)$

$= g(0)$.

fg $\quad e^{i\theta}(z+a)$

Inversion in circles

$z \overset{f}{\mapsto} \frac{1}{z}$ geometrically?

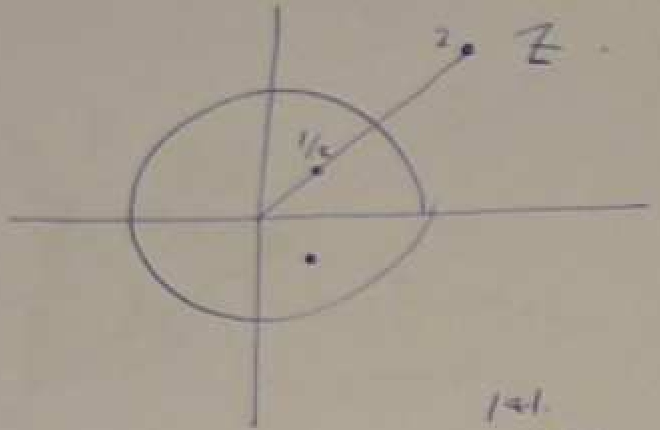
$|z| \qquad |f(z)| = \left|\frac{1}{z}\right| = \frac{1}{|z|}$

$\arg(z) \qquad \arg\left(\frac{1}{z}\right)$

$z\frac{1}{z} = 1 \leftarrow \arg(1) = 0$

$\arg(z) + \arg\left(\frac{1}{z}\right) = 0$

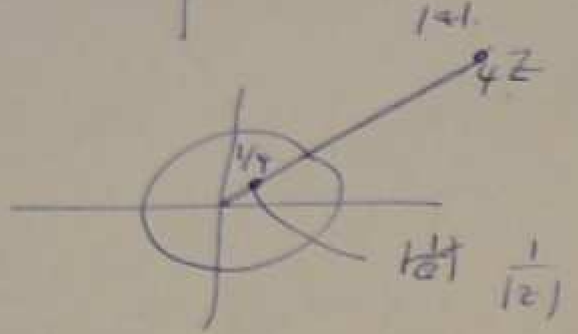
$\arg(z) = -\arg\left(\frac{1}{z}\right)$



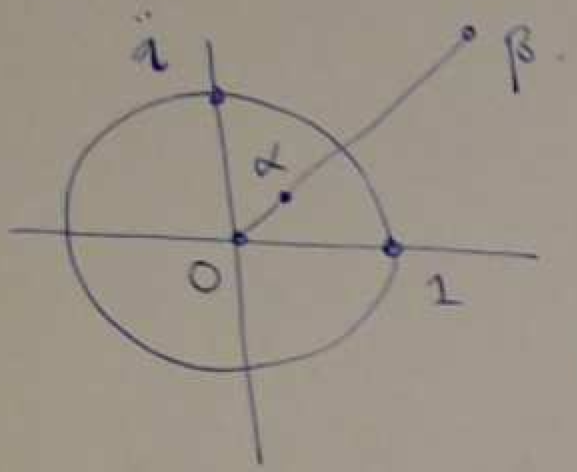
$g(z) = \frac{1}{\bar{z}}$

$\updownarrow$

inversion in circle



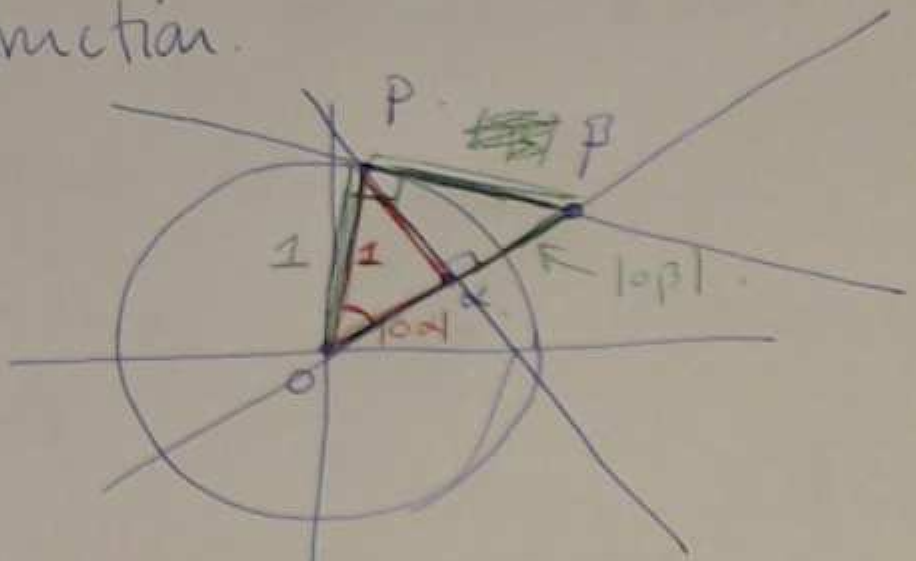
Geometric definition of inversion in a circle.



α lies on the straight line 0β.

$|0\alpha||0\beta| = 1$. (if circle has radius R) R^2

Construction.



1) construct line 0α

2) find $\perp$ line through α intersects circle at P

3) find tangent line at P.

4) β = tangent $\cap$ $\overline{0\alpha}$

$$\frac{|0\alpha|}{1} = \frac{1}{|0\beta|} \Rightarrow |0\alpha|\,|0\beta| = 1.$$

geometric maps.

$z \mapsto z+a$ translation

$z \mapsto e^{i\theta} z$ rotation

$z \mapsto \lambda z$ $\lambda \in \mathbb{R}^+$ $\neq 0$ is an dilation/~~expa~~ scaling

$z \mapsto \bar{z}$ reflection.

$z \mapsto \frac{1}{\bar{z}}$ inversion in a circle.

<u>Limits</u> recall: of sequence $(a_n)_{n\in\mathbb{N}}$

notation warning: $(a_n)_{n\in\mathbb{N}}$ (a_n) sequence.

$\{a_n\}$ ×.

$0,1,0,1,0,1,0,1,\ldots$ $\{0,1,0,1,\ldots\}$
$= \{0,1\}$.

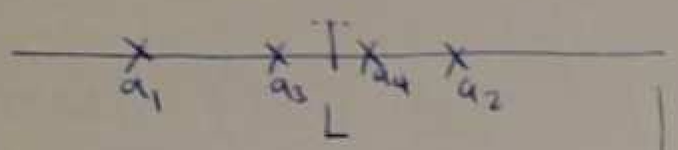
<u>Limits</u> recall (a_n) sequence of real numbers

then $\lim\limits_{n\to\infty} a_n = L$ if for all $\epsilon > 0$ there

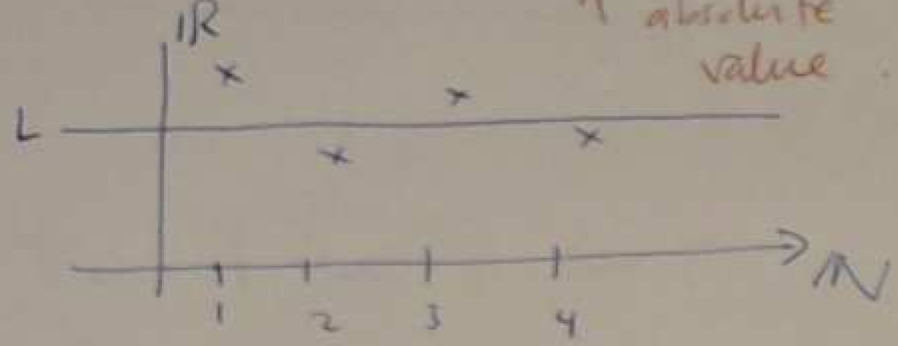
is an N s.t. for all $n \geq N$ $|L - a_n| < \epsilon$.

↑ absolute value.

$\mathbb{R}$



$\mathbb{C}$



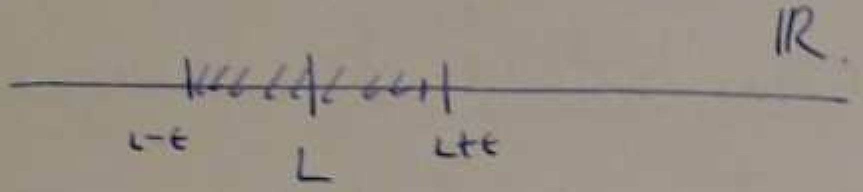
<u>Defn</u> (z_n) sequence of complex numbers

then $\lim\limits_{n\to\infty} z_n = L \in \mathbb{C}$ if for all $\epsilon > 0$ there is

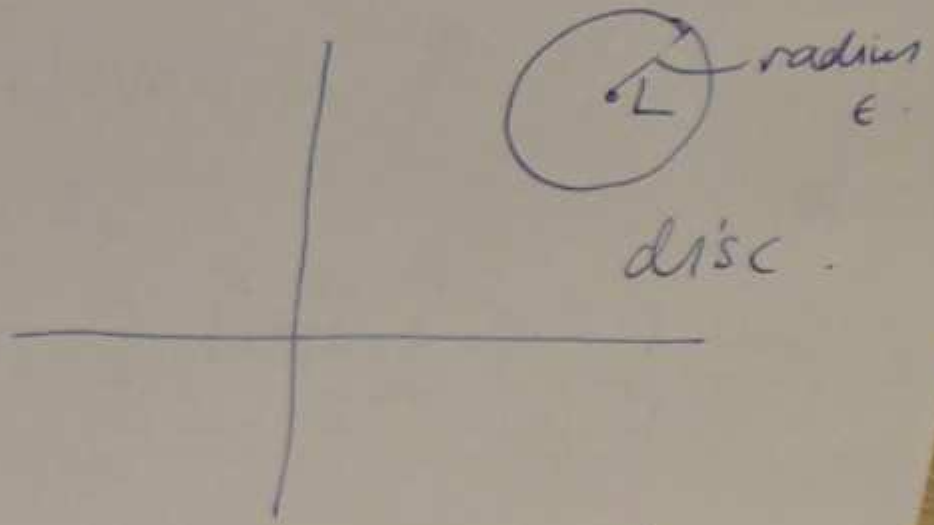
an N s.t. for all $n \geq N$ $|L - z_n| < \epsilon$.

↑ modulus

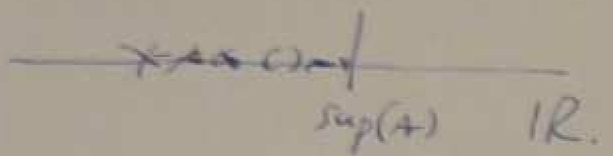
$|L - x| \leq \epsilon$ $\qquad \mathbb{R}$. $\qquad\qquad \mathbb{C}$. $\qquad |L - z| \leq \epsilon$.



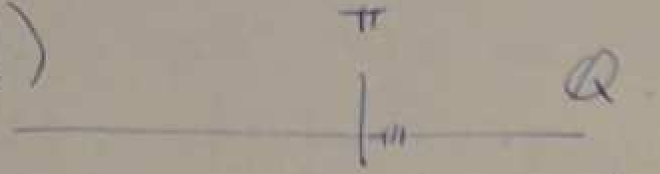
interval.



<u>recall</u> ($\mathbb{R}$) let $A \subseteq \mathbb{R}$ which is bounded above. Then A has a least upper bound called $\sup(A)$



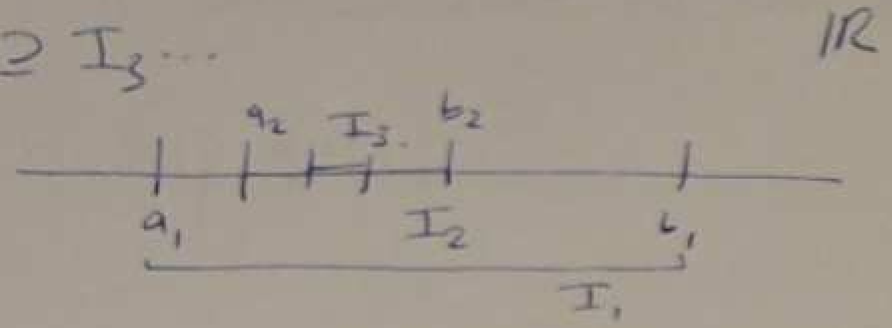
(not true in $\mathbb{Q}$)



$A = \{$ all rationals less than $\pi \}$.

recall **Thm** (Nested intervals) Let I_n be a sequence of 1) nested closed intervals in $\mathbb{R}$.

$I_n = [a_n, b_n]$. $\quad I_1 \supseteq I_2 \supseteq I_3 \cdots$



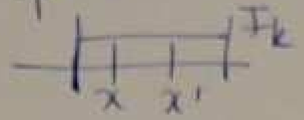
2) $\text{length}(I_n) \to 0$ as $n \to \infty$.

Then there is a unique real number x s.t.

$$\{x\} = \bigcap_{n=1}^{\infty} I_n$$

Proof uniqueness: spose there are two x, x' $\quad x \neq x'$

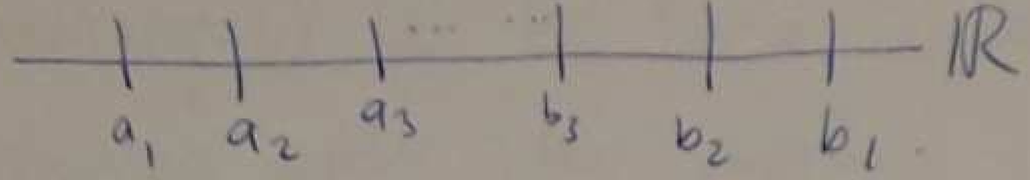
then $|x - x'|$



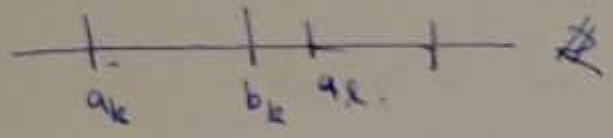
$\text{Length}(I_k) \geq |x - x'|$

$\not\to 0$ #

existence



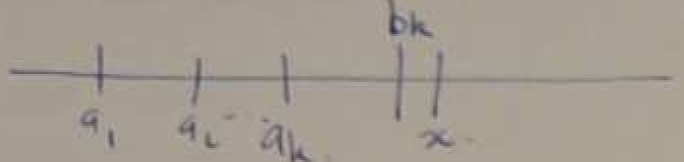
nested so the sequence (a_n) is bounded above by any b_k.



let $x = \sup\{a_n\}$

claim $x \in I_k$ for all k.

suppose not then



$b_k < x$

↑

↯ x least upper bound.

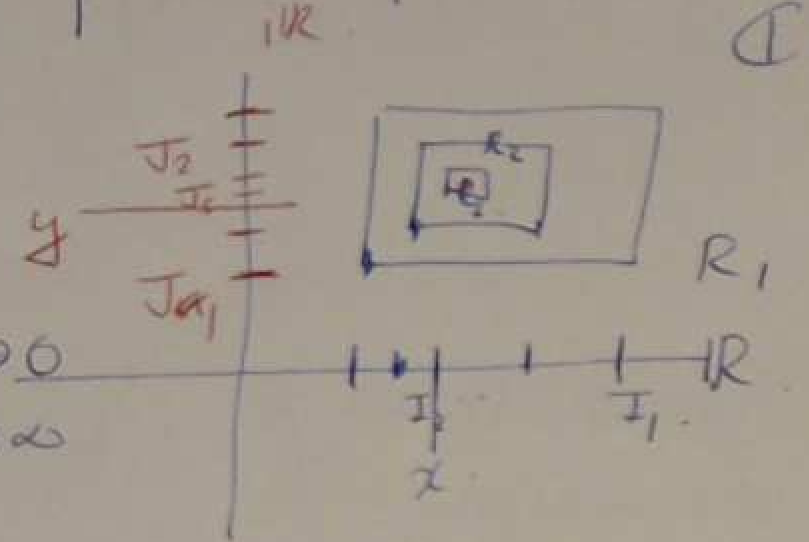
□

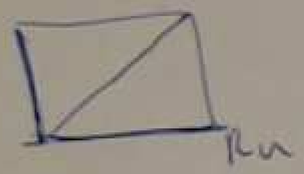
Thm (Nested rectangles) Let R_n be a sequence of nested rectangles with sides parallel to the coordinate axes.

$\mathbb{C}$.

1) $R_{n+1} \subseteq R_n$ for all n

2) length of diagonal of $R_n \to 0$ as $n \to \infty$





2) $\Rightarrow$ length of each side $\to 0$ as $n \to \infty$

project down to $\mathbb{R}$ $\rightsquigarrow$ get a sequence of nested intervals lengths $\to 0$ as $n \to \infty$. by previous, there is a unique $x \in \bigcap I_n$ similarly there is a unique $y \in \bigcap J_n$.

claim: $\overset{z}{x+iy}$ is the unique point in $\bigcap R_n$.

if there were two $z \neq z'$ $\quad |z-z'| > 0$ 

$\Rightarrow$ length diagonal $\not\to 0$ ⨳ $\square$.

Notational warning

limit of a sequence $\quad$ different $\quad$ a limit point of a sequence/set

example $\quad (z_n) = (0, 1, 0, 1, 0, 1, \dots)$

this sequence does not have a limit

but 0 and 1 are limit points of this sequence

<u>Defn</u> — A complex number L is a limit point for a sequence $(z_n)_{n \in \mathbb{N}}$ if for all $\epsilon > 0$ there are infinitely many terms z_n such that $|L - z_n| < \epsilon$.

<u>Examples</u>

$\left(\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots \right)$

↑ no limit but 0 and 1 are limit points

$(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \dots)$ no limit but 0 is a limit point

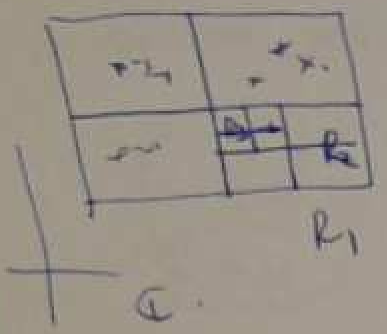
$(1, 2, 3, 4, 5, 6, \dots)$

no limit and no limit points.

Defn the sequence (z_n) is bounded if there is $M \in \mathbb{R}$ s.t. $|z_n| \le M$ for all n.

Thm (Bolzano–Weierstrass) Every bounded sequence in $\mathbb{C}$ has at least one limit point.

Proof. there is a rectangle R_1 containing all $(z_n)_{n \in \mathbb{N}}$.



divide into 4 congruent rectangles.
at least one contains infinitely many (z_n)
call it R_2 subdivide again...
this gives a nested sequence of rectangles $R_1 \supseteq R_2 \supseteq R_3 \dots$ each of which contains infinitely many (z_n).

by previous thm there is a unique $\{z\} = \bigcap_{n=1}^{\infty} R_n$

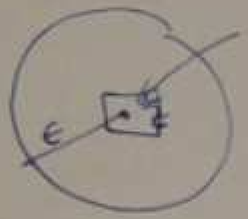
claim: z is a limit point of the (z_n).

Proof: suppose not, there is an $\epsilon > 0$ s.t.

$D_z = |z - z'| \leq \epsilon$ only contains finitely many points.

but length of diagonal of $R_n \longrightarrow 0$

so for n sufficiently large diagonal$(R_n) < \epsilon/100$



contains infinitely many (z_n) $\#$. $\square$.

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<u>Thm</u> / <u>Fact</u> for <u>limits</u> of sequences, the usual limit rules hold, if $\lim_{n\to\infty} z_n = L$ $\lim_{n\to\infty} z_n' = L'$

then $\lim_{n\to\infty}(z_n + z_n') = L + L'$

$\lim_{n\to\infty}(cz_n) = cL$

$\lim_{n\to\infty}(z_n z_n') = LL'$

$\lim_{n\to\infty}\left(\frac{z_n}{z_n'}\right) = \frac{L}{L'}$

<u>Proof</u> same as in $\mathbb{R}$ □

Joseph Maher

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(18)

<u>Thm</u> (Cauchy convergence ~~Thm~~ criteria) (z_n) converges iff for any $\epsilon > 0$ there is an N s.t. $|z_n - z_m| < \epsilon$ for all $n, m > N$.

<u>Warning</u> think about why $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ is <u>not</u> a counterexample.

<u>Proof</u> $\Rightarrow$ assume $z_n \to z$ as $n \to \infty$

given $\epsilon/2$ there is a N s.t. $|z - z_n| < \frac{\epsilon}{2}$ for all $n \geq N$.

$$|z_n - z_m| = |z_n - z - (z_m - z)| \overset{\text{triangle}}{\leq} \underbrace{|z_n - z|}_{\leq \epsilon/2} + \underbrace{|z_m - z|}_{\leq \epsilon/2} \leq \epsilon.$$

$\Leftarrow$ choose $\epsilon = 1$, with corresponding N.

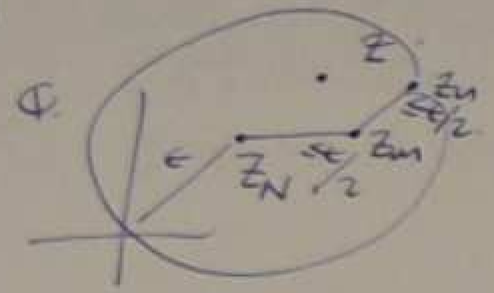
consider z_N then $|z_N - z_n| < 1$ for all $n \geq N$.

Joseph Maher

in particular, the sequence is bounded, so by Bolzano-Weierstrass there is a limit point z for the sequence $(z_n)_{n \geqslant N''}$. Now consider $\epsilon/2$ and the corresponding $N' = N(\epsilon/2)$

so $|z_n - z_m| < \epsilon/2$ for all $n, m \geqslant N'$ $\Big\}$ choose $N'' = \max\{N, N'\}$

$|z_N - z_m| < \epsilon/2$ for all $m \geqslant N$ $\Big\}$

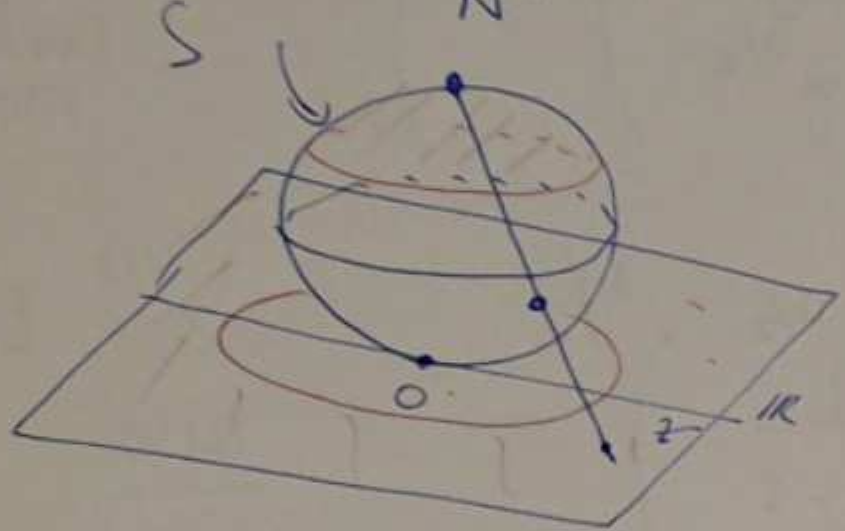


for all $m \geqslant N''$

$\Rightarrow$ $|z - z_N| \leq \epsilon$

$|z - z_m| \leq \underbrace{|z - z_N|}_{\epsilon} + \underbrace{|z_N - z_m|}_{\epsilon/2} = 3\epsilon/2$. $\square$

Riemann sphere.



stereographic projection.

$\mathbb{C}$. this gives a ~~map~~ bijection from $\mathbb{C} \longleftrightarrow S \setminus \{N\}$.

we can think of N as a "point at infinity" for $\mathbb{C}$.

Riemann sphere $\Leftrightarrow$ $\mathbb{C} \cup \{\infty\}$ extended complex plane.

<u>Defn</u> We say (z_n) approaches infinity $\lim_{n\to\infty} z_n = \infty$ if for any $M>0$ there is an N s.t. $|z_n| > M$ for all $n \geq N$.