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MTH 431 Complex Analysis.

Complex numbers.

$\mathbb{N}$.

motivations: counting numbers $1, 2, 3, 4, \ldots\ldots$

addition $a+b$ multiples ab.

solve equations: $x-2=3$.
$x=5$.

can't solve: $10-20=-10$.

add in negative numbers

$\ldots -2, -1, 0, 1, 2, 3, \ldots$ $\mathbb{Z}$.

$a+b$
$a-b$

<u>closed</u> under addition and subtraction

Joseph Maher

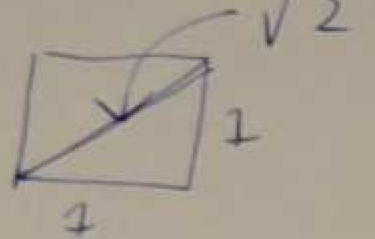
- can't solve: $3x = 2$ $\quad x = \frac{2}{3}$.

$\mathbb{Q}$ rationals | fractions $\left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \quad b \neq 0 \right\}$.

closed under addition/subtraction
multiplication/division.

- can't solve:

unit circle
perimeter 2π.

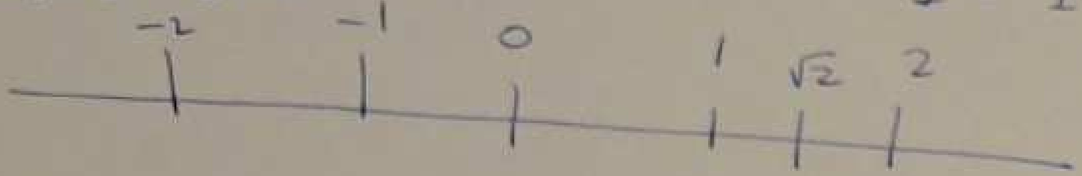


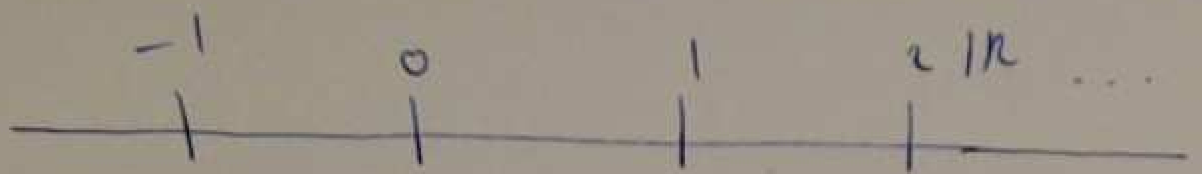
$\mathbb{R}$ reals. $\pi, \sqrt{2}, \ldots$ $\quad$ {infinite decimals}

$\frac{1}{2} = 0.50000\ldots$

$1 = 1.0000\ldots$

number line





$$\begin{aligned} &\mathbb{R} \longrightarrow \mathbb{R} \\ &\left.\begin{aligned} x &\longmapsto x+1 \\ x &\longmapsto x-1 \end{aligned}\right\} \quad x \longmapsto x+a \end{aligned}$$

translation by a.

$$\left.\begin{aligned} x &\longmapsto 2x \\ x &\longmapsto x/2 \end{aligned}\right\} \quad x \longmapsto ax \quad (a \neq 0)$$

expansions/contractions.

$x^2-2=0$

$x^2=2$.

$x=\pm\sqrt{2}$.

can't solve: $x^2+1=0$

solution: add new numbers.

i $\quad i^2=-1$

want this to satisfy same rules as other numbers.

$$2(i+1) = 2i+2$$

define $\mathbb{C} = \{ a+bi \mid a, b \in \mathbb{R} \}$.

Fact: this works.

Amazing fact: all polynomials have solutions in $\mathbb{C}$.

ax^2+bx+c: $\quad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$. $\qquad \sqrt{-4} = \sqrt{4}\sqrt{-1}$
$\qquad\qquad\qquad\qquad 2 \cdot i$

can also solve $\quad x^3 = -1$

$$x^5 - 4x + 1 = 0.$$

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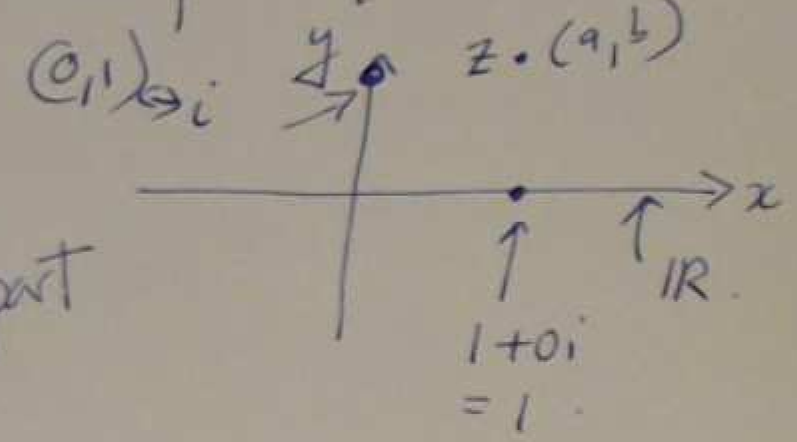
check

$$z \leftrightarrow \underbrace{a}_{\mathrm{Re}(z)} + \underbrace{b}_{\mathrm{Im}(z)} i$$

real part imaginary part

$$w \leftrightarrow c+di$$



addition: $z+w = a+bi+c+di$

$$= (a+c)+(b+d)i$$

subtraction: $z-w = a+bi-(c+di)$

$$= a+bi-c-di$$

$$= (a-c)+(b-d)i$$

$$\boxed{i^2 = -1}$$

multiplication: zw

$$(a+bi)(c+di) = ac + adi + bci + bd\underbrace{i^2}_{-1}$$

$$= (ac-bd) + (ad+bc)i$$

<u>division</u>: $\frac{z}{w}$

$w \neq 0$

$$\frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)}$$

$$= \frac{ac+bd+(bc-ad)i}{c^2+d^2-\cancel{cdi}+\cancel{cdi}}$$

$$= \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

$\mathbb{C}$

closed

add/sub

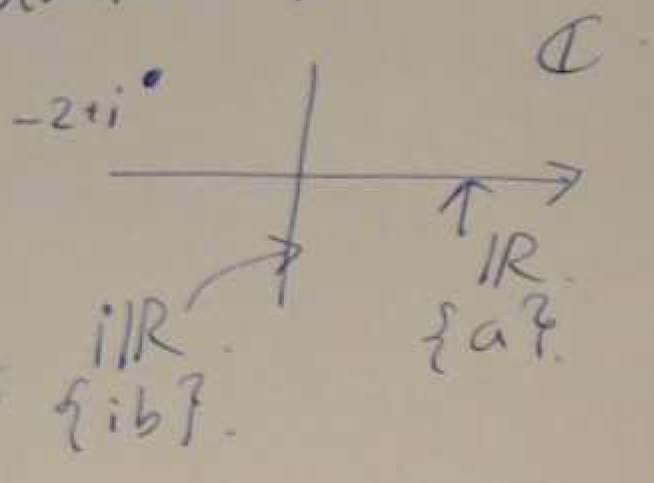
mult/divis

name for this

is a <u>field</u>

Notation $a+bi$ complex number

$b=0$ $a+0i = a$ real number $\mathbb{R} \subseteq \mathbb{C}$



Exercise check $\mathbb{C}$ is a field.

1) addition commutative $z+w = w+z$
associative $z+(w+t) = (z+w)+t$
inverses $z+(-z) = 0$ ← additive identity

2) multiplication: comm. $zw = wz$
ass. $(zw)t = z(wt)$.
inverse $z\frac{1}{z} = 1$ $(z \neq 0)$.

3) distribution $z(w+t) = zw+zt$.

Fact two complex numbers are the same iff they have the same real and imag. parts

$z = w$ $\qquad a+bi = c+di$

$\iff a = c$ and $b = d$.

Defn complex conjugates. $\quad z = a+bi$

complex conjugate $\longrightarrow \bar{z} = a - bi$

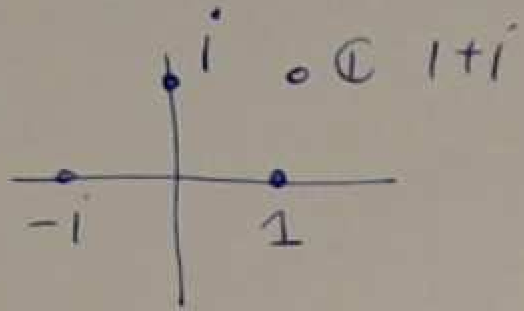
Observation $z\bar{z} = (a+bi)(a-bi) = a^2+b^2 \in \mathbb{R}$

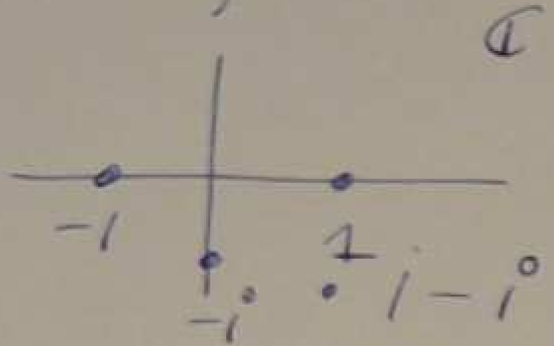
$\dfrac{z \cdot \bar{z}}{a^2+b^2} = 1 \qquad \dfrac{1}{z} = \dfrac{\bar{z}}{a^2+b^2} \leftarrow$ mult. inverse for $z \neq 0$.

$z \mapsto \bar{z}$

$a+bi \mapsto a-bi$

$$\left.\begin{array}{l} \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto \bar{z} \end{array}\right\} \text{ reflection in } \mathbb{R}.$$





<u>Fact</u> (no zero divisors)

$zw = 0 \Rightarrow$ at least one of z or w is zero

<u>Proof</u> suppose $z \neq 0$

$$\underbrace{\tfrac{1}{z} z}_{1} w = \tfrac{1}{z} 0 = 0 \Rightarrow w = 0 \quad \square$$

Observations $\overline{z+w} = \overline{z} + \overline{w}$ ① $\quad \overline{zw} = \overline{z}\,\overline{w}$

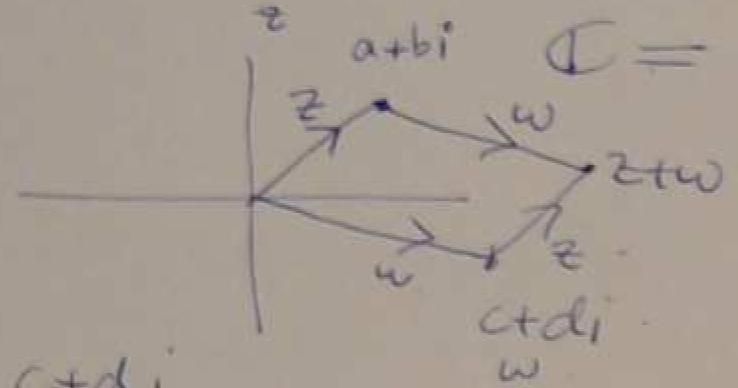
$z = w \iff \overline{z} = \overline{w}$.

check ① $\quad z = a+bi \quad w = c+di$.

$\overline{z+w} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$

$\overline{z}+\overline{w} = a-bi+c-di = (a+c)-(b+d)i$

Complex plane $\quad \mathbb{C} = \mathbb{R}^2$.



$z = a+bi \quad w = c+di$

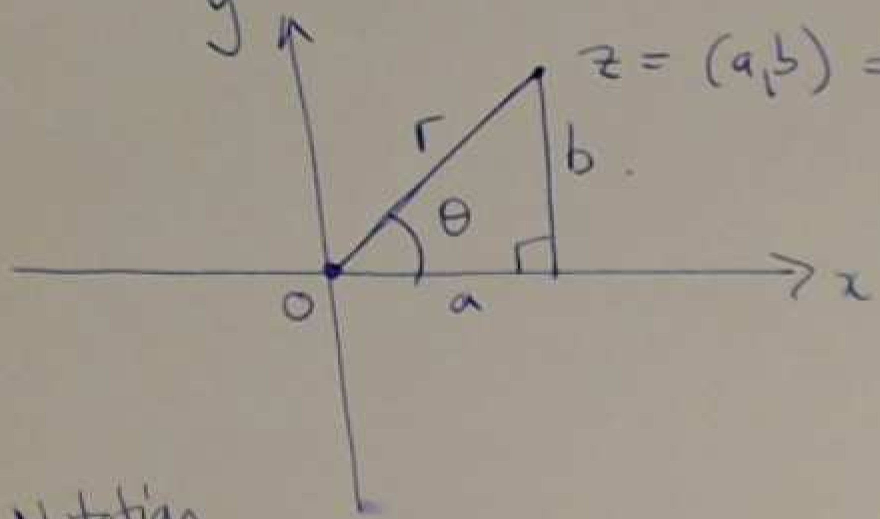
$z+w$?
$\shortparallel$
$(a+c)+(b+d)i$

$$\left.\begin{array}{l} \mathbb{C} \longrightarrow \mathbb{C}. \qquad a \in \mathbb{C}. \\ z \longmapsto z+a \end{array}\right\}$$ what is that geometrically? (11)

Polar coordinates.

$\mathbb{C} = \mathbb{R}^2$

rectangular/



$z = (a,b) = a+bi \leftarrow$ cartesian

(r,θ) polar coords.

$a = r\cos\theta$

$b = r\sin\theta$.

$a^2+b^2 = r^2$

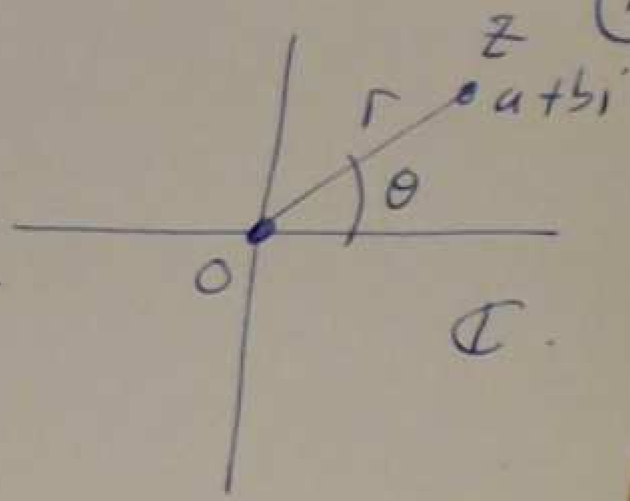
$\frac{b}{a} = \tan\theta$.

Notation

$r =$ modulus

$\theta =$ argument

Remarks.



notation modulus $|z| = \sqrt{z\bar{z}}$

$= \sqrt{a^2+b^2} = r$

$\arg(z) = \theta$.

warning $\arg(0)$ is not defined

$z = a+bi = r\cos\theta + r\sin\theta\, i$.

recall $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ (Euler).

$z = a+bi = r\cos\theta + r\sin\theta\, i = re^{i\theta}$.

$z+w$. $\quad a+bi+(c+di) = (a+c)+(b+d)i$ ✓.

zw. $\quad z = a+bi = re^{i\theta}$. $\quad w = c+di = se^{i\phi}$.

$$zw = re^{i\theta} se^{i\phi} = rs\, e^{i\theta} e^{i\phi} = rs\, e^{i(\theta+\phi)}$$

Remark multiplication multiplies moduli
adds arguments.

$$
\begin{aligned}
zw &= (a+bi)(c+di) \\
&= (r\cos\theta + r\sin\theta\, i)(s\cos\phi + s\sin\phi\, i) \\
&= rs\left[(\cos\theta\cos\phi - \sin\theta\sin\phi) + (\sin\theta\cos\phi + \cos\theta\sin\phi)i\right] \\
& \quad rs\left(\cos(\theta+\phi) + \sin(\theta+\phi)\, i\right)
\end{aligned}
$$

$z = re^{i\theta}$ $\qquad$ $z = a + bi = r\cos\theta + r\sin\theta\, i$

$$z^n = r^n e^{i\theta n} = r^n(\cos\theta n + i\sin\theta n)$$

$$\| $$

$$(r\cos\theta + r\sin\theta\, i)^n$$

$$r^n\left((\cos\theta + i\sin\theta)^n\right) = r^n(\cos\theta n + i\sin\theta n)$$

$r=1$

<u>$n=2$</u> $\quad \cos^2\theta - \sin^2\theta + 2\cos\theta\sin\theta\, i = \cos 2\theta + i\sin 2\theta$

<u>Recall:</u> $|z|$ modulus $\qquad |zw| = |z||w|$

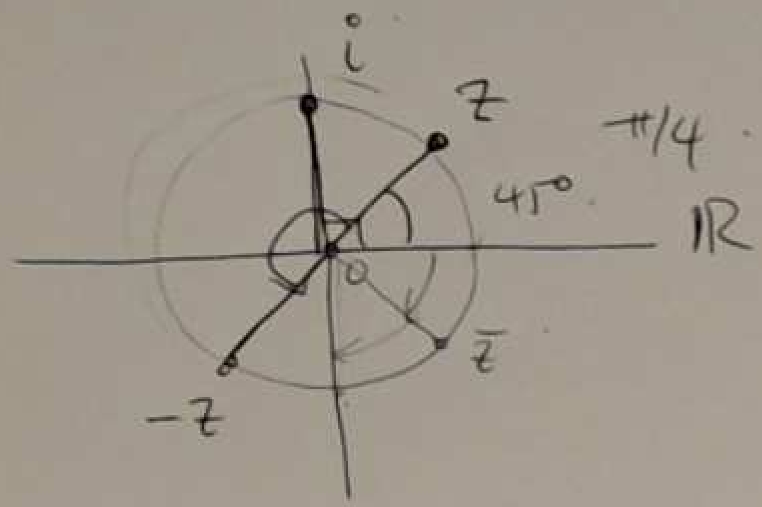
Exercise more generally $|z_1 z_2 \cdots z_n| = |z_1||z_2| \cdots |z_n|$

$$\arg(z_1 z_2 \cdots z_n) = \arg(z_1) + \arg(z_2) + \cdots + \arg(z_n)$$

Complex roots.

$\sqrt[n]{z}$ ← any complex number which raised to the nth power is equal to z.

Ex.



$i^2 = -1$

$\sqrt{i}$?

$z = \sqrt{i}$

$z^2 = i$

$|z^2| = |z|^2 = |i| = 1$

$z = e^{i\pi/4}$

$= \cos(\frac{\pi}{4}) + i\sin(\pi/4)$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

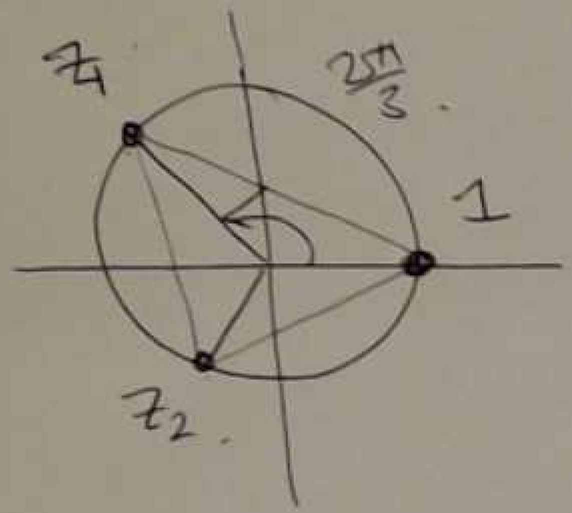
$\arg(z^2) = \arg(zz)$

$= \arg(z) + \arg(z)$

$= 2\arg(z)$.

$\arg(-z) = \pi + \frac{\pi}{4}$

$\arg(-z)^2 = 2(\pi + \frac{\pi}{4}) = \frac{\pi}{2}$



$$z^3 = 1 = 1 + 0i$$

$$z = 1.$$

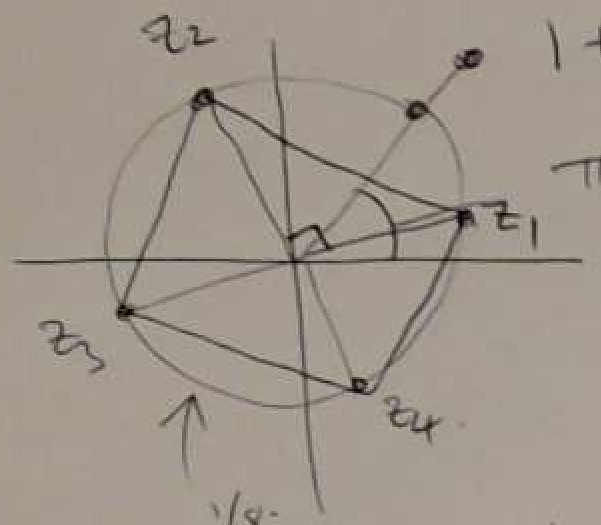
$$|1| = 1.$$

$$\arg(1) = 0 = 2\pi.$$

$$\sqrt[3]{1} \qquad \arg(z^3) = 3\arg(z) \qquad \arg(z) = \frac{2\pi}{3}.$$

$$z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right).$$

$$z_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right).$$



$$z = \sqrt[4]{1+i}$$

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}.$$

$$|z| = \sqrt[4]{\sqrt{2}} = 2^{1/8}$$

$$|z| = 2^{1/8}$$

$$4\arg(z) = \frac{\pi}{4}$$

z_1 $|z| = 2^{1/8}$

$$\arg(z) = \frac{\pi}{16}.$$

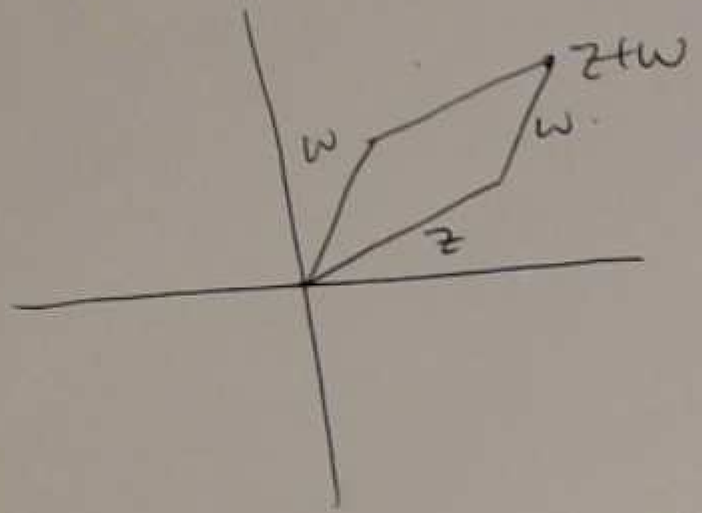
$$\arg(z_2) = \frac{\pi}{\cancel{4}\,16} + \underbrace{\frac{2\pi}{4}}_{\pi/2}$$

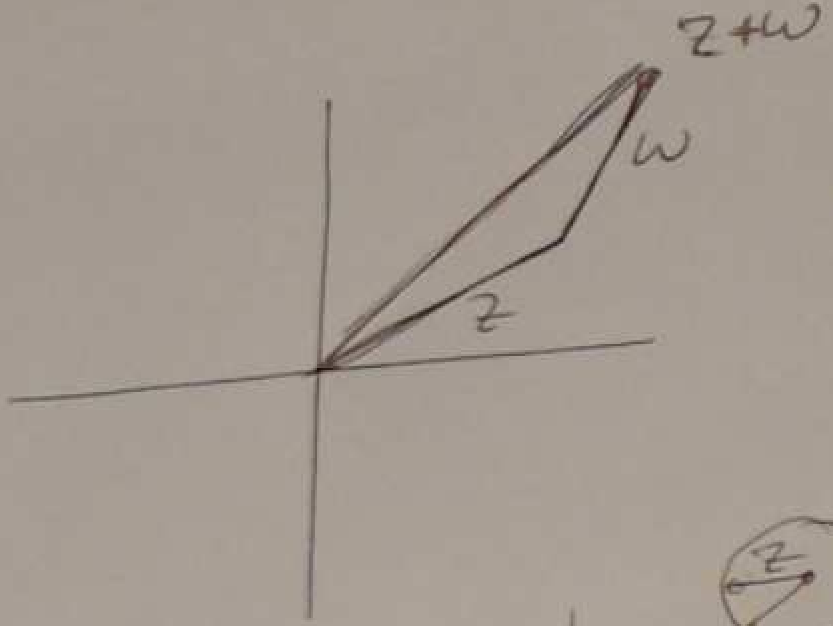
$$4\arg(z_2) = 4\left(\frac{\pi}{16} + \frac{2\pi}{4}\right) = \frac{\pi}{4} + \cancel{2\pi} \checkmark$$

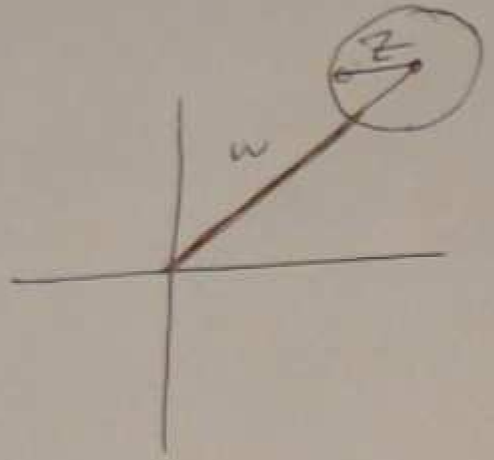
$$\arg(z_3) = \frac{\pi}{16} + \frac{4\pi}{4}.$$

$$\arg(z_4) = \frac{\pi}{16} + \frac{6\pi}{4}$$

$\mathbb{C}$.







triangle inequality

$$|\underset{\substack{\uparrow \\ w-z}}{z+w}| \leq |z| + |\underset{\substack{\uparrow \\ w-z}}{w}|.$$

variant

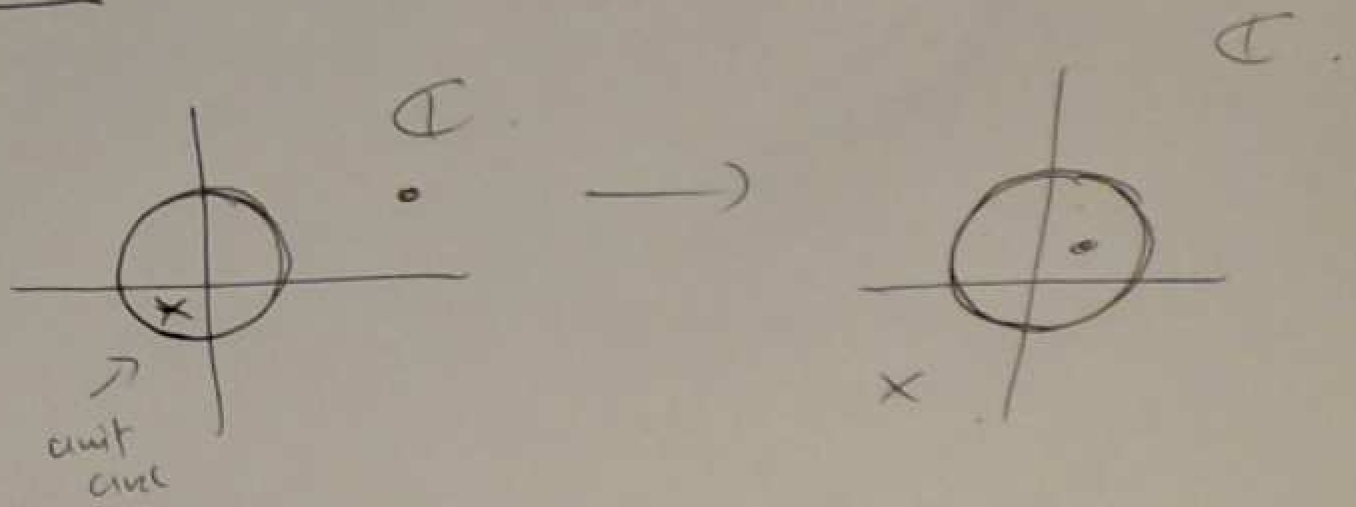
$$|w| = |\cancel{z}+w-\cancel{z}| \leq |z| + |w-z|.$$

$$|w-z| \geq \big||w| - |z|\big|. \qquad \checkmark \text{ symmetric in } z, w$$

inversion in circles

$z+w$.

$z \mapsto \frac{1}{z}$.



~~Ex.~~

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

$z^3 = 0$.

$\sqrt[3]{0}$!