Fibonacci numbers $\quad a_0=0, \quad a_1=1, \quad a_n = a_{n-1}+a_{n-2}$

$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$

set $\quad f(z) = \sum_{n\geq 0} a_n z^n, \quad$ check radius of convergence positive:

consider $\sum_{n=1}^{\infty} |a_n z^n| \leftarrow$ positive series, apply ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}z^{n}}{a_n z^n}\right|$

$$= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right||z| = \lim_{n\to\infty} \left|\frac{a_n+a_{n-1}}{a_n}\right||z| = \lim_{n\to\infty} \left|1+\frac{a_{n-1}}{a_n}\right||z|$$

$\leq 2|z|$ so converges for $|z|<\frac{1}{2}$.

Q: what is $f(z)$?

A: $\quad f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$

$\quad zf(z) = a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \dots$

$f + zf = a_0 + (a_0+a_1)z + (a_1+a_2)z^2 + (a_2+a_3)z^3 + \dots$

$f+zf = a_0=0 + a_2 z + a_3 z^2 + a_4 z^3 = \frac{f}{z}.$

$+\cancel{z}1 \quad + a_1\cancel{z}$

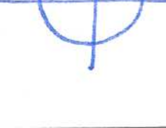
$f+zf+\frac{1}{\cancel{z}} = \frac{f}{z} \qquad f(1+z-\frac{1}{z}) = -\cancel{z}1 \qquad f(z) = \frac{-\cancel{z}1}{1+z-\frac{1}{z}} = \frac{z^{\cancel{z}}}{1-z-z^2}$

Q: how do we get a_n from $f(z)$?

A: $\quad \operatorname{Res}_{z=0} \frac{1}{z^n} f(z) = \operatorname{Res}_0 \frac{1}{z^n(1-z-z^2)} = \operatorname{Res}_0 \frac{1}{z^n}(a_0 + a_1 z + a_2 z^2 + \dots)$

$\frac{1}{z}$ term is $\quad \frac{a_{n-1}z^{n-1}}{z^n} = \frac{a_{n-1}}{z} \quad$ so $\operatorname{Res}_0 f = a_{n-1}$

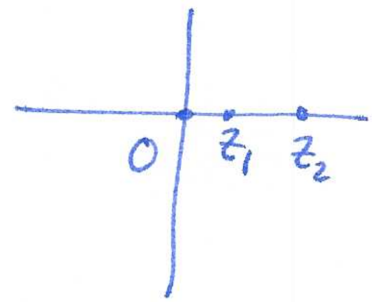
now: $\displaystyle\int_{\gamma_R} \frac{1}{z^n(1-z-z^2)}\,dz$



$\gamma_R : |z| = R.$

$$\left|\int_{\gamma_R} \frac{1}{z^n(1-z-z^2)}\,dz\right| \le 2\pi R\,\frac{1}{R^n}\,\frac{1}{R^2-R-1} \to 0 \text{ as } R\to\infty. \quad (n \geq 0).$$

so $\displaystyle\int_{\gamma_R} \frac{f(z)}{z^n}\,dz = 0 = \sum \operatorname{Res} \frac{f}{z^n}$.



$$z^2+z-1=0 \qquad z = \frac{-1\pm\sqrt{1+4}}{2} = z_1, z_2$$

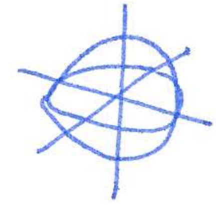
$$\operatorname*{Res}_{z_1} \frac{f}{z^n} = \lim_{z\to z_1} \frac{(z-z_1)}{-z^n(z-z_1)(z-z_2)} = \lim_{z\to z_1} -\frac{1}{z^n(z-z_2)} = -\frac{1}{z_1^n(z_1-z_2)}$$

$$\operatorname*{Res}_{z_2} f = \frac{-1}{z_2^n(z_2-z_1)}, \text{ so } \quad a_{n-1} - \frac{1}{z_1^n(z_1-z_2)} - \frac{1}{z_2^n(z_2-z_1)^n} = 0$$

$$a_{n-1} = \frac{1}{z_1^n(z_1-z_2)} + \frac{1}{z_2^n(z_2-z_1)}$$

Geometry Euclid's axioms:

1) any two points are connected by a straight line
2) any line segment can be extended to an infinite straight line
3) can draw circle with arbitrary centre and radius
4) all right angles are equal.
5) non parallel lines intersect. $\Leftarrow$ independent of first 4.



spherical geometry $S^2 = x^2+y^2+z^2=1$ in $\mathbb{R}^3$.

try: points $\leftrightarrow$ points.
lines $\leftrightarrow$ arcs of great circles
$\uparrow$ lines not unique

better: point $\leftrightarrow$ antipodal pair of points
lines $\leftrightarrow$ antipodal pairs on great circles $\Big\}$ satisfies first 4 $\to$ <u>no</u> parallel lines!

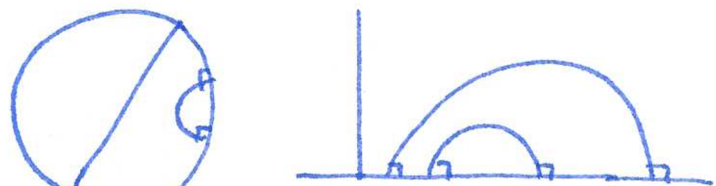
$\uparrow$ projective geometry

<u>hyperbolic geometry</u>: axioms 1–4) — infinitely many 'parallel' lines.

Two models: disc: lines are straight lines/circles $\perp$ to boundary

upper half space: "

angles = angle in the space.



← these two models give the same geometry, as there is a Möbius map taking one to the other and Möbius maps preserve angles and ~~take~~ preserve {circles/straight lines}.

<u>Defn</u> A <u>surface</u> is a space which is locally homeomorphic to a disc.

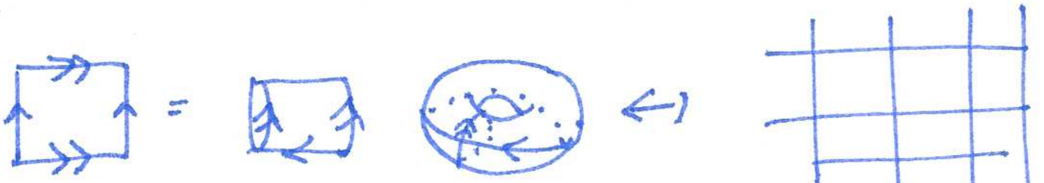
<u>Thm</u> (Classification of compact surfaces) Every compact, closed surface is one of:

S^2 T^2 $g=2$ S_2 $g=3$ S_3

<u>Fact</u>: each surface has a geometry associated with it.

S^2 ✓

T^2:



$T^2 = \mathbb{R}^2/\sim$

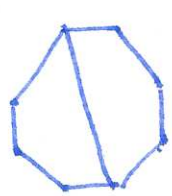
tiling of $\mathbb{R}^2$ by squares/parallelograms.

$(x,y) \sim (x+a, y+b)$
$a,b \in \mathbb{Z}$

<u>note</u>  also gives a torus?

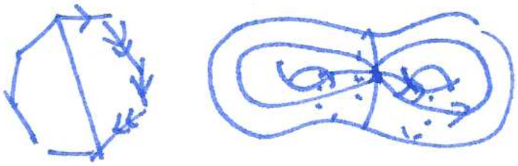
Q: are they the same? A: No.

S_3: hyperbolic geometry   Fact:



← glue opposite sides get S_2.

or



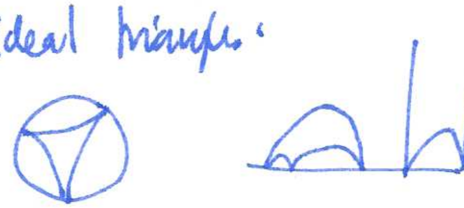
Q: do octagons tile the plane? A: No. (angles too big).

$360 \leftrightarrow 2\pi$.

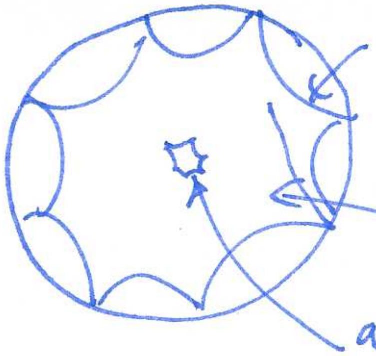
triangles in $\mathbb{H}^2$:

Σ angles $< \pi$

ideal triangles:



regular octagon



ideal regular octagon (angles = 0)

so are with angle $\frac{2\pi}{8}$ in the middle.

angles close to Euclidean angles.

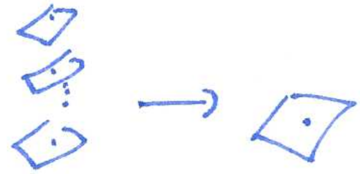
$\mathbb{H}^2$ is tiled by octagons.

$S_3 \longleftrightarrow$ 12gon w/ opp sides glued together $\leftrightarrow$ can also tile $\mathbb{H}^2 \ldots$

<u>Recall</u> : $f(z) = p(z) \leftarrow$ deg n poly $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. generically $n-1$

so inverse is multivalued

branched cover



compact closed surface, i.e. one of S^2, T^2, S_g

$\mathbb{C} \cup \{\infty\} = S^2$

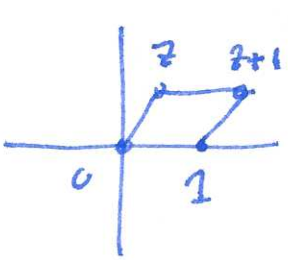
Q: how many geometric structures?

S^2: only one.

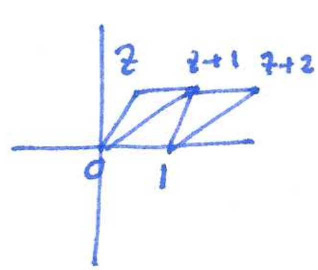
T^2: can be made from parallelogram  $\leftarrow$ can rotate / ~~reflect~~ rescale

so one edge is $[0,1] \subset \mathbb{R} \subset \mathbb{C}$.

$\mathbb{C}$.



$\longleftrightarrow$ equivalent to



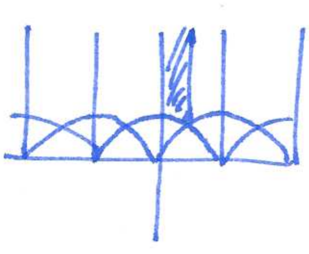
so $z \longleftrightarrow z+1$

could choose other side to rescale to length 1, horizontal $z \longleftrightarrow -\frac{1}{z}$

$w \mapsto \frac{w}{z}$ $\quad z \mapsto 1 \quad 1 \mapsto -\frac{1}{z}$

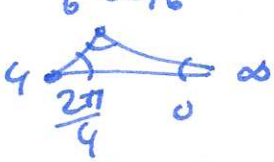
Fact: that's it.

Q: what is the space of Euclidean/complex structures on tori?



$z \mapsto z+1$

$z \mapsto -\frac{1}{z}$



Modular surface

$S^3 \setminus 3$ special pts