⑤ flow past a circular disc of radius R, assuming velocity at ∞ is $w_\infty = u_\infty + i v_\infty$. (68)

(assume flow solenoidal and irrotational).

?

$f(z)$ complex potential for the flow.

$f'(z)$ analytic in $|z|>R$. $f'(\infty) = \bar{w}_\infty$.

so $f'(z)$ has Laurent expansion

$$f'(z) = \bar{w}_\infty + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \quad \text{so} \quad f(z) = \bar{w}_\infty z + c_1 \ln(z) - \frac{c_2}{z} - \frac{c_3}{2z^2} - \cdots$$

stream function $\psi(r,\theta) = \operatorname{Im} f(z)$. use polars $z = re^{i\theta}$

set $c_1 = a_1 + ib_1$ $c_2 = a_2 + ib_2$, etc.

$$\psi(r,\theta) = a_1\theta + b_1 \ln(r) + \frac{a_2 + r^2 u_\infty}{r}\sin\theta - \frac{b_2 + r^2 v_\infty}{r}\cos\theta + \cdots$$

$$+ \frac{a_3}{2r^2}\sin 2\theta - \frac{b_3}{2r^2}\cos\theta + \cdots$$

let Γ be circle $|z|=R$ ← must be streamline for flow.

so $\psi(R,\theta) = \text{const}$. ← can satisfy this if

$a_1 = 0$

$a_2 = -R^2 u_\infty$

$b_2 = -R^2 v_\infty$.

$a_3 = b_3 = \cdots = 0$.

gives $$f'(z) = \bar{w}_\infty + \frac{ib_1}{z} - \frac{R^2 w_\infty}{z^2}$$

$$f(z) = ib_1 \ln(z) + \bar{w}_\infty z + \frac{R^2 w_\infty}{z}. \qquad b_1 \in \mathbb{R}.$$

K

circulation around Γ: $$K = \operatorname{Re}\int_\Gamma f'(z)\,dz = \operatorname{Re}\int_{|z|=r} f'(z)\,dz \quad \text{for any } r>R.$$

$$K = \operatorname{Re}\int_{|z|=r}\left\{\bar{w}_\infty + \frac{ib_1}{z} - \frac{R^2 w_\infty}{z^2}\right\}dz = -2\pi b_1, \quad \text{so } b_1 = \frac{-K}{2\pi}$$

so $f'(z) = \bar{w}_\infty + \dfrac{K}{2\pi i z} - \dfrac{R^2 w_\infty}{z^2}$

wlog assume $w_\infty = u_\infty > 0$

(do a rotation).

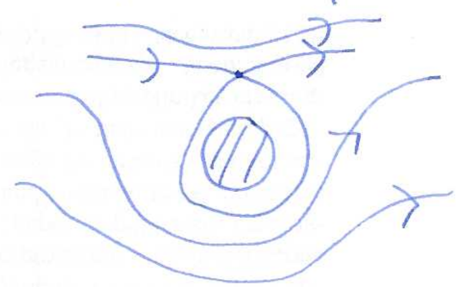
$$f(z) = \frac{K}{2\pi i}\ln(z) + \bar{w}_\infty z + \frac{R^2 w_\infty}{z}$$

find stagnation points: $f'(z) = 0$, set ie $u_\infty + \dfrac{K}{2\pi i z} - \dfrac{R^2 u_\infty}{z^2} = 0$

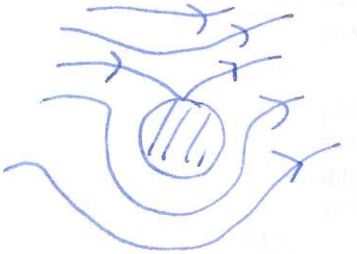
$$z^2 + \frac{K}{2\pi i u_\infty} z - R^2 = 0$$

sols: $z_{1,2} = \dfrac{iK}{4\pi u_\infty} \pm \sqrt{R^2 - \left(\dfrac{K}{4\pi u_\infty}\right)^2}$.

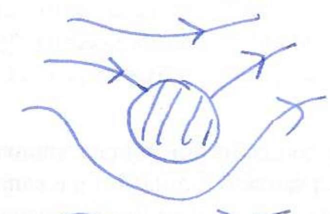
- $|K| > 4\pi R u_\infty$: z_1, z_2 purely imaginary, but $z_1 z_2 = -R^2$, so only one lies outside $|z| = R$. get:



- $|K| = 4\pi R_\infty$: one stagnation point on $|z| = R$

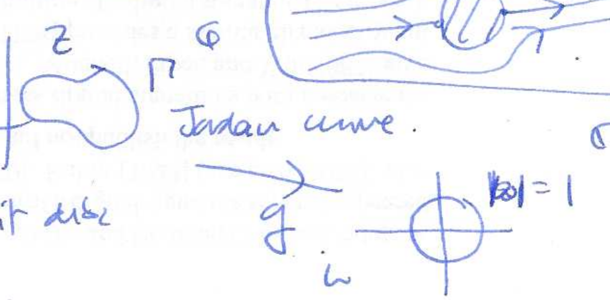


- $|K| < 4\pi R u_\infty$: two stagnation points on $|z| = R$.



note if $K = 0$ (no circulation).

Q: spose object is not a circle/~~cylinder~~.



Fact: there is a unique conformal map that takes $\mathbb{C} \setminus \text{int}(\Gamma)$ to $\mathbb{C} \setminus$ Int of unit disc s.t. $g(\infty) = \infty$, and $g'(\infty) \in \mathbb{R}_{>0}$.

Laurent expansion of g is $g(z) = cz + c_0 + \dfrac{c_1}{z} + \cdots$

solution for $|w| = 1$ is $\Phi(w) = \dfrac{K}{2\pi i}\ln(w) + \bar{A} w + \dfrac{A}{w}$.

so $$f(z) = \Phi(g(z)) = \frac{K}{2\pi i} \ln(g(z)) + \bar{A} g(z) + \frac{A}{g(z)}$$

want $\bar{w}_\infty = f'(\infty) = \Phi'(\infty) g'(\infty) = \bar{A} c$ s.t. $A = \frac{w_\infty}{c} = \frac{w_\infty}{g'(\infty)}$

so $$f(z) = \frac{K}{2\pi i} \ln(g(z)) + \frac{\bar{w}_\infty}{g'(\infty)} g(z) + \frac{w_\infty}{g'(\infty) g(z)}$$

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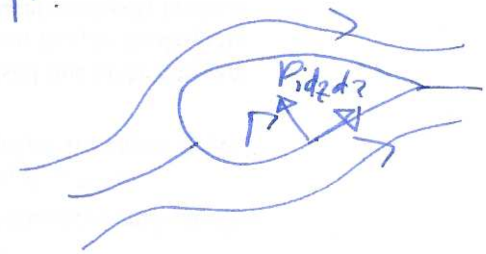
find the force exerted on the cylinder of cross section Γ by flow past the cylinder w/ $w_\infty = u_\infty + i v_\infty$ at infinity

A: $F = -\rho K \dot{w}_\infty i$ ρ = density of fluid (constant)
K circulation around Γ.



let $P(x,y)$ = pressure at (x,y)

Bernoulli's law: $P + \frac{1}{2}\rho |w|^2$ constant
along flow lines, and hence on Γ, so $P = A - \frac{1}{2}\rho |w|^2$ for some $A \in \mathbb{R}_{>0}$

$$P i dz = A i dz - \frac{1}{2} \rho i |w|^2 dz.$$

total force $F = X + iY$ on $\Gamma = \int_\Gamma P i \, dz = A i \int_\Gamma dz - \frac{1}{2}\rho i \int_\Gamma |w|^2 dz$

so $$F = -\frac{1}{2}\rho i \int_\Gamma |w|^2 dz$$

w tangent to Γ, so $w = \overline{f'(z)} = |w| e^{i\theta}$ $\theta = \arg dz$

$$|w| = \overline{f'(z)}\, e^{-i\theta}$$

$$F = -\frac{1}{2}\rho i \int_\Gamma [\overline{f'(z)}]^2 e^{-2i\theta} dz = -\frac{1}{2}\rho i \int_\Gamma \overline{[f'(z)]^2 dz}$$

$$\bar{F} = X - iY = \frac{1}{2}\rho i \int_\Gamma (f'(z))^2 dz$$

$$\bar{F} = \frac{1}{2}\rho i \int_{|z|=r} \left\{ \bar{w}_\infty + \frac{k}{2\pi i z} + \frac{c_1}{z^2} + \cdots \right\}^2 dz$$

$$= \frac{1}{2}\rho i \, \frac{2k\bar{w}_\infty}{2\pi i} \int_C \frac{dz}{z} = \frac{1}{2}\rho i \, \frac{2k\bar{w}_\infty}{2\pi i} \, 2\pi i = \rho k \bar{w}_\infty i$$

$$F = -\rho k w_\infty i.$$

§15.3 Electrostatics

electric field $\leftrightarrow$ vector field, gives force experienced by a unit positive charge

stationary: independent of time.

plane-parallel: same in all parallel planes
and has no components $\perp$ to planes.

$$E(x,y) = E_x(x,y) + iE_y(x,y)$$

Maxwell equations: $\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = 0 \qquad \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} = 4\pi\rho$

where $\rho =$ charge density

G simply connected domain, with $\rho = 0$ in G.

Green's Thm: $\displaystyle\int_C E_x dx + E_y dy = 0 \qquad \int_C -E_y dx + E_x dy = 0$

so the electrostatic field is both irrotational and solenoidal in G,

so both $-E_x dx - E_y dy$ and $-E_y dx + E_x dy$ are exact, i.e.

there are real functions $\phi(x,y)$ $\psi(x,y)$ s.t.

$$\left.\begin{aligned} -E_x dx - E_y dy &= d\phi \\ -E_y dx + E_x dy &= d\psi \end{aligned}\right\} \qquad \begin{aligned} E_x &= -\frac{\partial\phi}{\partial x} & E_y &= -\frac{\partial\phi}{\partial y} \\ E_y &= -\frac{\partial\psi}{\partial x} & E_x &= \frac{\partial\psi}{\partial y} \end{aligned}$$