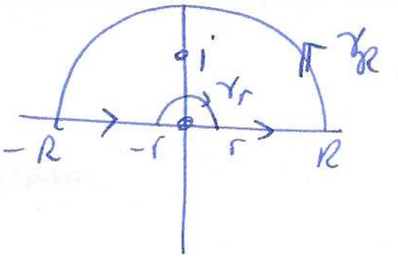
Multiple valued functions

Example $\displaystyle\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx$



Let $f(z) = \dfrac{\ln(z)}{(z^2+1)^2}$ ← need to choose branch

$-\pi < \operatorname{Im}\ln z = \arg z < \pi.$

$$\operatorname*{Res}_i f(z) = \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2 \ln z}{(z^2+1)^2} = \lim_{z\to i} \frac{d}{dz} \frac{\ln(z)}{(z+i)^2} = \lim_{z\to i} \frac{(z+i)^2 \frac{1}{z} - \ln(z).2(z+i)}{(z+i)^4}.$$

$$= \lim_{z\to i} \frac{\frac{-4}{i} - \frac{\pi}{2}\cdot 4i}{(2i)^4} = \frac{+4i - 2\pi}{16} = \frac{\pi+2i}{8i}$$

$$\int_{-R}^{r} + \int_{\gamma_r} + \int_r^R + \int_{\gamma_R} = 2\pi i\left(\frac{\pi+2i}{8i}\right) = \frac{\pi^2 i}{4} - \frac{\pi}{2}.$$

γ_R : $z = Re^{i\theta}$ $\quad |\ln z| = |\ln R + i\theta| \le \sqrt{(\ln R)^2 + \pi^2} \le 2\ln R$ for $R >> 0$.

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \frac{2\ln R}{(R^2-1)^2}\pi R \to 0 \text{ as } R \to \infty.$$

$$\gamma_r : \left|\int_{\gamma_r} f(z)\,dz\right| \le \frac{2\ln(1/r)}{(1-r^2)^2}\pi r \to 0 \text{ as } r \to 0.$$

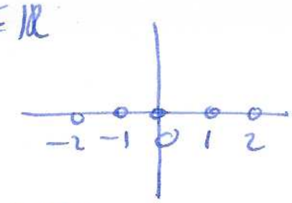
$$\int_{-\infty}^{0} \frac{\ln x}{(x^2+1)^2}\,dx + \int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx = \frac{\pi^2 i}{4} - \frac{\pi}{2}. \qquad \ln(-x) = \ln x + \pi i$$

$\shortparallel$

$$\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx + \pi i \int_0^\infty \frac{dx}{(x^2+1)^2}\,dx.$$

$$\text{So}\quad 2\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx + \pi i \int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi^2 i}{4} - \frac{\pi}{2}.$$

Example consider $\pi\cot(\pi z)$ ← simple poles at $\mathbb{Z} \subseteq \mathbb{R}$

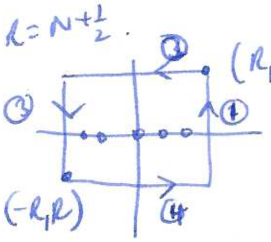


residues: $\lim\limits_{z\to n} (z-n)\,\pi \dfrac{\cos(\pi z)}{\sin(\pi z)} = \pi(-1)^n \lim\limits_{z\to n} \dfrac{z-n}{\sin \pi z} \overset{l'H}{=} \pi(-1)^n \lim\limits_{z\to n} \dfrac{1}{\pi\cos(\pi z)}$

$= \dfrac{\pi(-1)^n}{\pi(-1)^n} = 1.$

now consider $f(z) = \dfrac{\pi\cot(\pi z)}{z^2}$ ← has residue $\dfrac{1}{n^2}$ at $\pm n$ $(\neq 0)$.

$R = N + \frac{1}{2}$.



$\underbrace{\int_{\gamma_R} f(z)\,dz}_{} = \operatorname*{Res}_0 f(z) + 2\sum_{i=1}^{N} \dfrac{1}{n^2}$

claim $\to 0$ as $R \to \infty$.

② ④ claim $|\cot(z)| \le 2$ if $|\operatorname{Im}(z)| \ge 1$. use $|\cos^2 z| = \cos^2 x + \sinh^2 y$

$|\sin^2 z| = \sin^2 x + \sinh^2 y$

$|\cot z|^2 \le \dfrac{1+\sinh^2 y}{\sinh^2 y} \le 2$. so $\left|\int_{②④}\right| \le \int \dfrac{2}{|z|^2}\,dz \underset{\sim \frac{R}{R^2}}{\to 0}$ as $R \to \infty$.

① ③ ~~$\le \frac{1}{R^2}$ checked~~ $\int_{-1}^{1}$ $x = (N+\frac{1}{2})$ so $\cos(\pi R) = 0$, $\sin(\pi R) = \pm 1$ so $\left|\int_{①③}\right| \le \left|\dfrac{2}{|z|^2}\right| R \le \dfrac{1}{R} \to 0$

Finally: Calculate $\operatorname*{Res}_0 \dfrac{\pi\cot(\pi z)}{z^2}$ ← can use $\lim\limits_{z\to 0} \dfrac{d}{dz} z^3 \dfrac{\pi\cot(\pi z)}{z^2}$.

or

$\dfrac{\pi\cot(\pi z)}{z^2} = \dfrac{\pi\cos(\pi z)}{z^2\sin(\pi z)} = \dfrac{\pi\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)}{z^2\left(\pi z - \frac{\pi^3 z^3}{3!} + \cdots\right)} = \dfrac{1}{z^3}\,\dfrac{1 - \frac{\pi^2 z^2}{2!} + \cdots}{1 - \frac{\pi^2 z^2}{3!} + \cdots}$

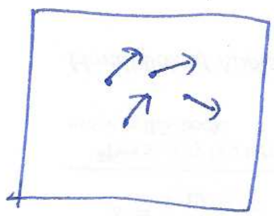
$= \dfrac{1}{z^3}\left(1 - \dfrac{\pi^2 z^2}{2!} + \cdots\right)\left(1 - \left[\dfrac{\pi^2 z^2}{3!} + \cdots\right] + \left[\dfrac{\pi^2 z^2}{3!} + \cdots\right]^2 + \cdots\right).$

$\dfrac{1}{z}$ term: $-\dfrac{\pi^2}{2} + \dfrac{\pi^2}{6} = -\dfrac{\pi^2}{3}$. so $0 = -\dfrac{\pi^2}{3} + 2\sum \dfrac{1}{n^2}$.

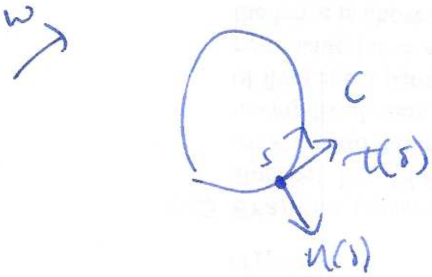
$\Rightarrow \sum \dfrac{1}{n^2} = \dfrac{\pi^2}{6}.$

§15.1 Fluid dynamics

(2d case)



at each point the fluid is moving with some velocity, i.e. vector valued function $w : \mathbb{R}^2 \to \mathbb{R}^2$, $w : \mathbb{C} \to \mathbb{C}$. $w(x,y) = u(x,y) + iv(x,y)$



$z(s)$ ↓ unit speed $0 \le s \le l$.
parameterization, C has length l.

$\tau(s)$, unit tangent vector to $C(s)$
$n(s)$ unit normal vector

$$\tau(s) = z'(s) = \frac{dx}{ds} + i\frac{dy}{ds}$$ ← unit length as $z(s)$ unit speed parameterization

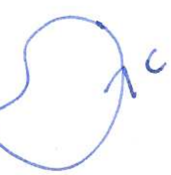
$$n(s) = \frac{1}{i} z'(s) = \frac{dy}{ds} - i\frac{dx}{ds}.$$

can write flow vector as (tangent part) + (normal part)
w_τ w_n.

w_τ component = scalar product of w and τ. $w_\tau = u\frac{dx}{ds} + v\frac{dy}{ds}$.

w_n components : scalar product of w and n $w_n = -v\frac{dx}{ds} + u\frac{dy}{ds}$

assume : $w = u + iv$ cts, differentiable, partial derivatives exist + are cts...



$$\int_C w_\tau \, ds = \int_C (u+iv)_\tau \, ds = \int_C u\,dx + v\,dy \quad (1)$$

<u>circulation around C</u> ↗

↙ <u>flux through C</u>

$$\int_C w_n \, ds = \int (u+iv)_n \, ds = \int -v\,dx + u\,dy \quad (2)$$

if ① = 0 for all closed curves C flow is <u>irrotational</u>

if ② = 0 <u>solenoidal</u>

consider

$$\int_C \bar{w}\, dz = \int_C \overline{u+iv}\,(dx+idy) = \int_C u\,dx + v\,dy + i\int_C -v\,dx + u\,dy$$

so: ① $\int_C u\,dx + v\,dy = \mathrm{Re} \int_C \bar{w}\, dz$

② $\int_C -v\,dx + u\,dy = \mathrm{Im} \int_C \bar{w}\, dz$

<u>Thm</u> A continuously differentiable flow $u+iv$ defined in a simply connected domain G is irrotational and solenoidal iff

$u + iv = \overline{f'(z)}$ where $f(z)$ is analytic in G

^ <u>complex potential</u> of the flow.

<u>Proof</u> suppose all $\int_C$ ①, ② are zero.

then $u\,dx + v\,dy$ and $-v\,dx + u\,dy$ are exact differentials, i.e.

there are real functions $\phi(x,y)$, $\psi(x,y)$ s.t.

$u\,dx + v\,dy = d\phi \qquad -v\,dx + u\,dy = d\psi$

$\Rightarrow \quad u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \qquad -v = \frac{\partial \psi}{\partial x} \quad u = \frac{\partial \psi}{\partial y}$

so ϕ, ψ satisfy CR equations so $f(z) = \phi + i\psi$ is analytic in G.

$f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = u - iv.$ □

- $u, -v$ conjugate harmonic functions.

① $\int_C w_\tau \, ds = \int_C u\,dx + v\,dy = \mathrm{Re} \int_C f'(z)\,dz$

② $\int_C w_n \, ds = \int_C -v\,dx + u\,dy = \mathrm{Im} \int_C f'(z)\,dz$.

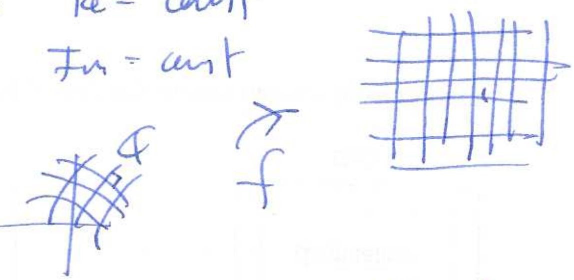
ϕ velocity potential ψ stream function

$\phi(x,y) = \text{const}$ $\psi(x,y) = \text{const}$ ← contours.

equipotentials streamlines.

the map $f(z)$ is conformal, except where $f'(z) = 0$ stagnation points.

$f(z)$ takes $\phi(x,y) = \text{const}$ to $\mathrm{Re} = \text{const}$
$\psi(x,y) = \text{const}$ $\mathrm{Im} = \text{const}$ $\mathbb{C}$.

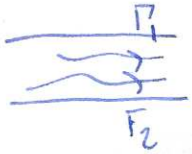


check: $\phi(x,y) = \text{const}$ $\psi(x,y) = \text{const}$

$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u\,dx + v\,dy = 0$ $\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\,dx + u\,dy = 0$.

velocity is tangential to the streamlines
↑ these are the actual trajectories of the fluid.

Fact If domain G has $\partial G = \Gamma$

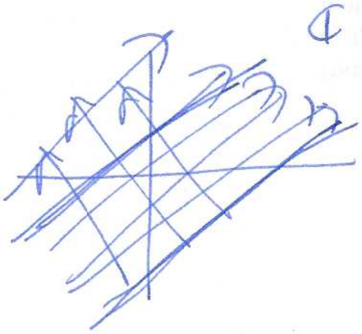


then each curve in ∂G must be a streamline for the flow (flow can't cross boundary), i.e. $\psi(x,y) = \text{const}$ on ∂G.
$\mathrm{Im}\, f(z)$.

Examples ① $f(z) = \alpha z$ ← flow on whole plane with uniform velocity

$\overline{f'(z)} = \bar{\alpha}$

if $\alpha = a+ib$ $\quad \phi(x,y) = ax - by \quad \psi(x,y) = bx + ay$

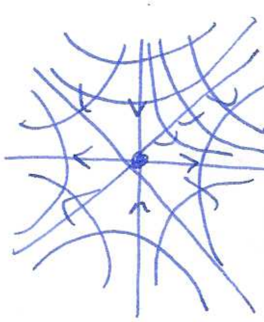


$\mathbb{C}$ $\bar{\alpha}$. this also works for uniform flow in a strip with two parallel sides parallel to $\bar{\alpha}$.

② $f(z) = z^2$. flow on whole of $\mathbb{C}$ (non-uniform velocity).

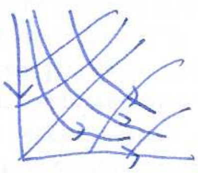
$\overline{f'(z)} = 2\bar{z}$ $\quad \phi(x,y) = x^2 - y^2 \quad \psi(x,y) = 2xy$

equipotentials $\qquad$ stream lines



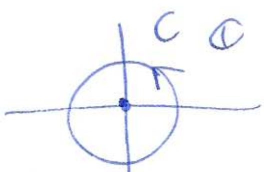
0 is stagnation point

note: this also gives the flow in one quadrant



← i.e. flow around a corner.

④ circular flows: $\mathbb{C} \setminus \{0\}$. $\quad f(z) = \frac{K}{2\pi i} \ln(z) \quad K \in \mathbb{R}$.



$\int_C w_\tau \, ds = K \qquad \int_C w_n \, ds = 0$

circulation K $\qquad$ flux $= 0$

$f(z) = \frac{\mu}{2\pi} \ln(z) \qquad \mu \in \mathbb{R}$.

$\int w_\tau \, ds = 0 \qquad \int_C w_n \, ds = \mu$

circulation $= 0$. $\qquad$ flux $= \mu$