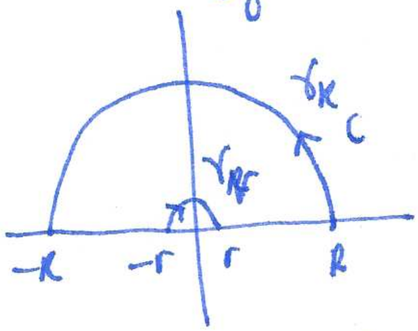
Example $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ consider $f(z) = \frac{e^{iz}}{z}$ ← pole at $z=0$!



$f(z)$ is analytic on and in C.

$$\int_C f(z)\,dz = 0 = \int_{-R}^{R}$$

$$\int_C f(z)\,dz = 0 = \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{\gamma_r} \frac{e^{iz}}{z}\,dz + \int_r^R \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz = 0$$

take limit as $r \to 0$ $R \to \infty$:

$$\int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx + \underbrace{\lim_{r\to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\,dz}_{②} + \int_0^\infty \frac{e^{ix}}{x}\,dx + \underbrace{\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\,dz}_{①} = 0.$$

① $$\int_{\gamma_R} \frac{e^{iz}}{z}\,dz \underset{\substack{\text{integrate}\\ \text{by parts}}}{=} \frac{e^{iz}}{iz}\Big|_{-R}^{R} + \int_{\gamma_R} \frac{e^{iz}}{iz^2}\,dz$$

$$= \frac{e^{iR} + e^{-iR}}{iR} + \frac{1}{i}\int_{\gamma_R} \frac{e^{iz}}{z^2}\,dz$$

$$\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,dz\right| \le \left|\frac{e^{iR}+e^{-iR}}{iR}\right| + \left|\frac{1}{i}\int_{\gamma_R} \frac{e^{iz}}{z^2}\,dz\right| \le \frac{2}{R} + \frac{\pi R}{R^2} \to 0 \text{ as } R\to\infty.$$

so $$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\,dz = 0.$$

② $$\lim_{r\to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\,dz$$

$\frac{e^{iz}}{z}$ has Laurent expansion at $z=0$:

$$\frac{1}{z}\left(1 + iz + \dots\right) = \frac{1}{z} + P(z)$$

$$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{iz}}{z}\,dz = \underbrace{\lim_{r\to 0}\int_{\gamma_r}\frac{dz}{z}}_{} + \underbrace{\lim_{r\to 0}\int_{\gamma_r}P(z)\,dz}_{0} = \lim_{r\to 0}\int_{\gamma_r}\frac{dz}{z}$$

$$\int_{\gamma_r}\frac{1}{z}\,dz = \int_{\pi}^{0}\frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = -\pi i$$

$|P(z)| \le M$ in a nbd. of $z=0$

$$\left|\int_{\gamma_r}P(z)\,dz\right| \le M\pi r \to 0 \text{ as } r\to 0.$$

so $$\lim_{r\to 0}\int_{\gamma_r}\frac{e^{iz}}{z}\,dz = -\pi i.$$

take imaginary parts!

finally:

$$\int_{-\infty}^{0}\frac{\sin x}{x}\,dx + \int_{0}^{\infty}\frac{\sin x}{x}\,dx = \pi$$

so $$\int_{0}^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

<u>Group actions</u>. $\mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$.

$\mathbb{C}$.

$z \mapsto az+b$

$a\bar{z}+b$ $\quad |a| = 1 \quad a = e^{i\theta}$

$b \in \mathbb{C}$.

Q: what sort of isometries of $\mathbb{R}^2$ are there?

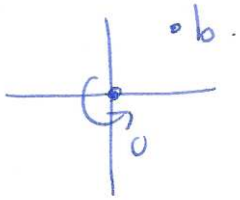
A: translations, rotations, reflections, glide reflections.

<u>can conjugate subgroups</u>:

Rotations about zero $H = \{ z \mapsto az \}, \; |a| = 1\}$ subgroup.

rotation about b. conjugate by $g: z \mapsto z+b$. $\quad z \mapsto z-b$

$z \mapsto e^{i\theta}(z-b)$.



gHg^{-1} ← another subgroup. $z \mapsto e^{i\theta}z - e^{i\theta}b + \cancel{e^{i\theta}}b$.

$\mathrm{stab}(0)$ ← contains H! also contains. $z \mapsto e^{i\theta}\bar{z}$ ← reflection in a line thru 0.

<u>check</u>. $f(z) = e^{i\theta}z + b \quad f(0) = 0: \quad 0 = b$.

$f(z) = e^{i\theta}\bar{z} + b \quad f(0) = 0: \quad 0 = b$.

unordered pairs of points (x, y). send x to 0. $z \mapsto z - x$.

$(x,y) \mapsto (0, y-x)$ Q: where can we send $y-x$? can use any element of $\mathrm{stab}(0)$.

classification (up to conjugacy).

<u>claim</u>: any isometry is conjugate to one of:

	$z \mapsto z$	
translation	$z \mapsto z+b$	← no fixed pts (unless $b=0$).
rotation	$z \mapsto e^{i\theta}z$	← one fixed pt.
reflection	$z \mapsto \bar{z}$	
glide reflection	$z \mapsto \bar{z} + r \in \mathbb{R}$	

$f: z \mapsto e^{i\theta} z + b$. $\quad \theta = 0 \Rightarrow$ translation ✓.

$\theta \neq 0$ look for fixed point $\quad z = e^{i\theta} z + b$. $\quad z(1-e^{i\theta}) = b$.

$$z = \frac{b}{1-e^{i\theta}} \in \mathbb{C}.$$

$f := \tau \rho \tau^{-1} \qquad \rho = \tau^{-1} f \tau$.

$$z \overset{\tau}{\longmapsto} \cancel{e^{i\theta} + b}\; z + \frac{b}{1-e^{i\theta}} \overset{f}{\longmapsto} e^{i\theta}\left(z + \frac{b}{1-e^{i\theta}}\right) + b = e^{i\theta} + \frac{be^{i\theta}}{1-e^{i\theta}} + b.$$

$$\overset{\tau^{-1}}{\longmapsto} e^{i\theta} + \frac{be^{i\theta}}{1-e^{i\theta}} + b - \frac{b}{1-e^{i\theta}} = e^{i\theta} + b\left[\frac{e^{i\theta}}{1-e^{i\theta}} + 1 - \frac{1}{1-e^{i\theta}}\right] = e^{i\theta}.$$

$f: z \mapsto e^{i\theta}\bar{z} + b$. $\quad$ conjugate by a rotation $\quad z \mapsto e^{i\phi}$.

$$z \mapsto e^{-i\phi} z \mapsto e^{i\theta}(e^{i\phi}\bar{z}) + b \mapsto e^{i\phi} e^{i\theta} e^{i\phi}\bar{z} + e^{i\phi} b.$$

$$e^{i(\theta+2\phi)}\bar{z} + e^{i\phi} b.$$

choose $\phi = -\frac{\theta}{2}$.

conjugate to $\quad z \mapsto \bar{z} + b' = \bar{z} + x + iy$.

conjugate by $\quad z \mapsto z + ci$

$$z \mapsto z - ci \mapsto \bar{z} + ci + x + iy \mapsto \bar{z} + ci + x + iy + ci.$$

$$= \bar{z} + x + i(y + 2c) \qquad \text{choose } c = -\frac{y}{2}.$$

conjugate to $\quad z \mapsto \bar{z} + x \in \mathbb{R}$.

($x = 0$ reflection).

<u>Residue at infinity</u> $\quad \operatorname{Res}_\infty f = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$.

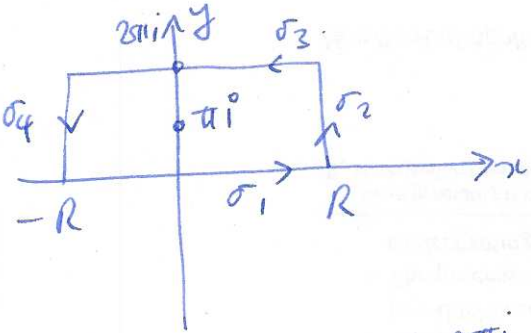
$$\left[f\left(\tfrac{1}{z}\right) d\left(\tfrac{1}{z}\right) = -f\left(\tfrac{1}{z}\right)\frac{1}{z^2}\right].$$

with this defn. $\quad \sum_{z_i} \operatorname{Res}_{z_i} = 0$

if f has isolated singularities

$z_1, \dots z_n$ (including ∞).

Example $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx \quad (0<a<1)$. consider $\displaystyle f(z) = \frac{e^{az}}{1+e^{z}}$.



residue at πi (l'H)

$$\operatorname*{Res}_{\pi i} f(z) = \lim_{z\to\pi i} \frac{(z-\pi i)e^{az}}{1+e^{z}} = \lim_{z\to\pi i} \frac{e^{az}+aze^{az}-a\pi i e^{az}}{e^{z}}$$

$$= \frac{e^{a\pi i} + \cancel{a\pi i e^{a\pi i}} - \cancel{a\pi i e^{a\pi i}}}{-1} = -e^{a\pi i}.$$

so $\displaystyle\int_C f(z)\,dz = -2\pi i e^{a\pi i}$

$$\int_{\sigma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ax}}{1+e^{x}}\,dx \qquad \int_{\sigma_3} f(z)\,dz = \int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx = -e^{2a\pi i}\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}\,dx$$

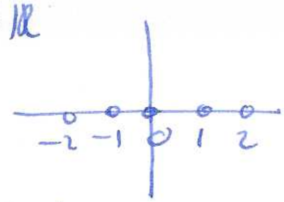
$$\sigma_2: \quad |f(z)| = \left|\frac{e^{a(R+iy)}}{1+e^{R+iy}}\right| \le \frac{e^{aR}}{e^{R}-1} \to 0 \text{ as } R\to\infty.$$

$$\sigma_4: \quad |f(z)| = \left|\frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\right| \le \frac{e^{-aR}}{1-e^{-R}} \to 0 \text{ as } R\to\infty.$$

so $$\left(1-e^{2a\pi i}\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = -2\pi i e^{a\pi i}$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{-2\pi i\, e^{a\pi i}}{1-e^{2a\pi i}} = \frac{\pi}{\sin a\pi}.$$

<u>Example</u> consider $\pi\cot(\pi z)$ ← simple poles at $\mathbb{Z} \subseteq \mathbb{R}$

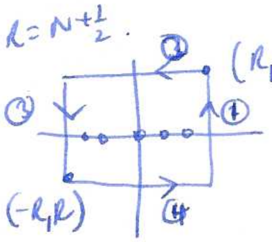


residues: $\lim_{z\to n} (z-n)\, \pi \frac{\cos(\pi z)}{\sin(\pi z)} = \pi(-1)^n \lim_{z\to n} \frac{z-n}{\sin \pi z} \overset{l'H}{=} \pi(-1)^n \lim_{z\to n} \frac{1}{\pi \cos(\pi z)}$

$= \frac{\pi(-1)^n}{\pi(-1)^n} = 1.$

now consider $f(z) = \frac{\pi\cot(\pi z)}{z^2}$ ← has residue $\frac{1}{n^2}$ at $\pm n \; (\neq 0)$.

$R = N + \frac{1}{2}$.



$$\underbrace{\int_{\gamma_R} f(z)\,dz}_{} = \operatorname{Res}_0 f(z) + 2\sum_{i=1}^{N} \frac{1}{n^2}$$

claim → 0 as $R \to \infty$.

②④ <u>claim</u> $|\cot(z)| \leq 2$ if $|\mathrm{Im}(z)| \geq 1$. use $|\cos^2 z| = \cos^2 x + \sinh^2 y$

$|\sin^2 z| = \sin^2 x + \sinh^2 y$

$|\cot z|^2 \leq \frac{1+\sinh^2 y}{\sinh^2 y} \leq 2$. so $\left|\int_{②④}\right| \leq \int \frac{2}{|z|^2}\,dz \to 0$ as $R \to \infty$. $\sim \frac{R}{R^2}$

①③ ~~$\varepsilon \frac{1}{R^2}$ tired~~ $x = (N+\frac{1}{2})$ so $\cos(\pi R) = 0$, $\sin(\pi R) = \pm 1$ so $\left|\int_{①③}\right| \leq \left|\frac{2}{|z|^2}\right| R$. $\leq \frac{1}{R} \to 0$

<u>finally</u>: calculate $\operatorname{Res}_0 \frac{\pi\cot(\pi z)}{z^2}$ ← can use $\lim_{z\to 0} \frac{d}{dz} z^3 \frac{\pi\cot(\pi z)}{z^2}$.

<u>or</u> $\frac{\pi\cot(\pi z)}{z^2} = \frac{\pi\cos(\pi z)}{z^2 \sin(\pi z)} = \frac{\pi\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)}{z^2\left(\pi z - \frac{\pi^3 z^3}{3!} + \cdots\right)} = \frac{1}{z^3} \frac{1 - \frac{\pi^2 z^2}{2!} + \cdots}{1 - \frac{\pi^2 z^2}{3!} + \cdots}$

$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{2!} + \cdots\right)\left(1 - \left[\frac{\pi^2 z^2}{3!} + \cdots\right] + \left[\frac{\pi^2 z^2}{3!} + \cdots\right]^2 + \cdots\right)$.

$\frac{1}{z}$ term: $-\frac{\pi^2}{2} + \frac{\pi^2}{6} = -\frac{\pi^2}{3}$. so $0 = -\frac{\pi^2}{3} + 2\sum \frac{1}{n^2}$.

$\Rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6}$.