**Corollary** If $f(z)$ is univalent in $G$, then $f(z)$ is conformal for all $z \in G$.

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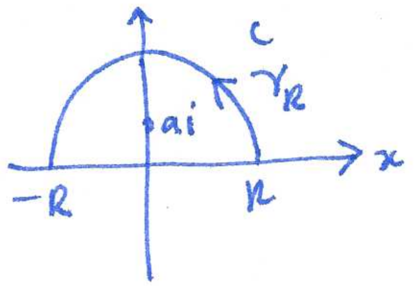
## Evaluating real integrals

**Example** $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3}\,dx \quad a>0$ consider $f(z) = \dfrac{1}{(z^2+a^2)^3}$




singular pts of $f(z)$: $\pm ai \leftarrow$ poles of order 3.

$$\operatorname*{Res}_{ai} f(z) = \frac{1}{2!}\lim_{z\to ai} \frac{d^2}{dz^2} \frac{(z-ai)^3}{(z^2+a^2)^3}$$

$$= \frac{1}{2}\frac{d^2}{dz^2}\frac{1}{(z+ai)^3}\Bigg|_{z=ai} = \frac{1}{2}\frac{3\cdot 4}{(2ai)^5} = \frac{3}{16a^5 i}$$

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ai} f(z) = \frac{3\pi}{8a^5}$$

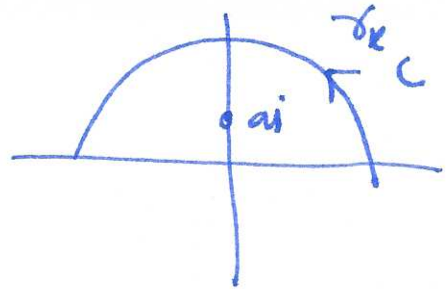
note: $\dfrac{1}{|z^2+a^2|} = \dfrac{1}{|z^2-(-a^2)|} \le \dfrac{1}{\big||z|^2-|a|^2\big|} = \dfrac{1}{R^2-a^2}$ for $z \in \gamma_R$

so $\left|\int_{\gamma_R} f(z)\,dz\right| \le \dfrac{\pi R}{(R^2-a^2)^3}$, so $\displaystyle\lim_{R\to\infty}\int_{\cancel{-R}\,\gamma_R}^{\cancel{R}} f(z)\,dz = 0$.

therefore $\displaystyle\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \cancel{\lim}\int_{-\infty}^{\infty} f(x)\,dx = \frac{3\pi}{8a^5}$

so $\displaystyle\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^3}\,dx = \frac{3\pi}{8a^5} \quad (a>0)$

Example $\displaystyle\int_0^\infty \frac{\cos x}{x^2+a^2}\,dx \quad (a>0)$




consider $f(z) = \dfrac{e^{iz}}{z^2+a^2}$

on $\gamma_R$ : $|e^{iz}| = e^{-y} \le 1$ as $y \geq 0$.

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \frac{\pi R}{R^2-a^2}, \text{ so } \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = 0$$

singular points of $f(z)$ at $\pm ai$ ← simple pole.

$$\operatorname*{Res}_{ai} f(z) = \lim_{z\to ai} (z-ai)\frac{e^{iz}}{z^2+a^2} = \left.\frac{e^{iz}}{z+ai}\right|_{z=ai} = \frac{e^{-a}}{2ai}$$

so $$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx + \int_{\gamma_R} f(z)\,dz = 2\pi i\,\frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}.$$

so $$\int_{-R}^{R} \frac{\cos x}{x^2+a^2}\,dx + \underbrace{\operatorname{Re}\int_{\gamma_R} f(z)\,dz}_{\to 0 \text{ as } R\to\infty} = \frac{\pi e^{-a}}{a}.$$

so $$\lim_{R\to\infty}\int_{-R}^{R} \frac{\cos x}{x^2+a^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos x}{a^2+x^2}\,dx = \frac{\pi e^{-a}}{a}$$

so $$\int_0^\infty \frac{\cos x}{a^2+x^2}\,dx = \frac{\pi e^{-a}}{2a} \quad (a>0)$$