the ~~...~~ residue of $\frac{d}{dz}(\ln f(z)) = \frac{f'(z)}{f(z)}$ at a.

- suppose a is a zero of order α, i.e. $f(z) = c_\alpha (z-a)^\alpha + c_{\alpha+1}(z-a)^{\alpha+1} + \dots$, $c_\alpha \neq 0$

so $f'(z) = \alpha c_\alpha (z-\alpha)^{\alpha-1} + \dots$

so $$\frac{f'(z)}{f(z)} = \frac{\alpha c_\alpha (z-\alpha)^{\alpha-1} + \dots}{c_\alpha (z-\alpha)^\alpha + \dots} = \frac{1}{z-\alpha} \frac{\alpha + (\alpha+1)\frac{c_{\alpha+1}}{c_\alpha}(z-a) + \dots}{1 + \frac{c_{\alpha+1}}{c_\alpha}(z-a) + \dots}$$

$$= \frac{\alpha}{z-\alpha} + c_0' + c_1'(z-a) + \dots$$

so $\operatorname{Res}_a \frac{f'(z)}{f(z)} = \alpha$, i.e. the order of the zero.

- suppose b is a pole of order β for $f(z)$.

then
$$\left.\begin{aligned} f(z) &= \frac{c_{-\beta}}{(z-b)^\beta} + \frac{c_{-\beta+1}}{(z-b)^{\beta-1}} + \dots \\ f'(z) &= -\frac{\beta c_{-\beta}}{(z-b)^{\beta}} + -\frac{(\beta-1)c_{-\beta+1}}{(z-b)^{\beta-2}} \dots \end{aligned}\right\} \frac{f'(z)}{f(z)} =$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z-b}\left(\frac{-\beta c_{-\beta} - (\beta-1)c_{-\beta+1}(z-b) - \dots}{c_{-\beta} + c_{-\beta+1}(z-b) + \dots}\right) = \frac{-\beta}{z-b} + c_0' + c_1'(z-b) + \dots$$

So $\operatorname{Res}_b \frac{f'(z)}{f(z)} = -\beta$.

<u>Thm</u> C piecewise smooth Jordan curve, $f(z)$ analytic in and on C, except for poles $b_1, \dots, b_m$ and $f(z)$ has zeros $a_1, \dots, a_n$ inside C, but not on C. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k$$

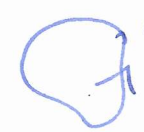
α_k = order of zero a_k
β_k = order of pole β_k.

Proof singular points of $\frac{f'(z)}{f(z)}$ inside C are exactly the poles and zeros of $f(z)$.

apply residue formulas. $\square$.

Let $\left.\begin{array}{l} N = \#\text{ of zeros} \\ P = \#\text{ of poles} \end{array}\right\}$ counted according to multiplicity

then $N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{d}{dz}\big(\ln(f(z))\big)\,dz$.



$= \frac{1}{2\pi i}\int_C d\ln(f(z)) = \frac{1}{2\pi i}\underbrace{\Delta_C \ln(f(z))}_{\text{change in } \ln f(z) \text{ as } z \text{ goes round } C}$

$\ln f(z) = \ln|f(z)| + i\arg(f(z))$ ← real part doesn't change. so

$\Delta_C \ln f(z) = i\,\Delta_C \arg(f(z))$, i.e. $N - P = \frac{1}{2\pi}\Delta_C \arg f(z)$.

Geometric interpretation: winding number

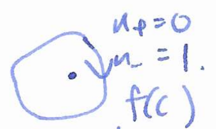
$f: \gamma \to \mathbb{C}$

$C\gamma \mapsto f(\gamma)$

$n_+ =$ number of times go round in +ve direction

$n_- =$ number of times go round in −ve direction

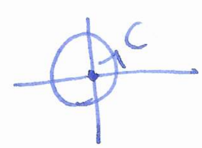
$\boxed{v = n_+ - n_-}$.

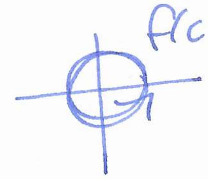


so $\Delta_C \arg f(z) = 2\pi v$, i.e. $N - P = v$

in image (w-plane).

Example $z \mapsto z^2$.





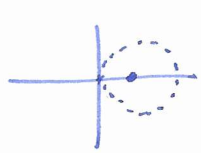
Thm (Rouché) f, g analytic inside and on C piecewise Jordan curve, and suppose $|f(z)| > |g(z)|$ on C

then $f(z)$ and $f(z) + g(z)$ have same number of zeros in C.

Proof $f(z) \neq 0$ on C, so $\Delta_C \arg(f(z) + g(z)) = \Delta_C \arg\left(f(z)\left(1 + \frac{g(z)}{f(z)}\right)\right)$

$= \Delta_C \arg(f(z)) + \Delta_C \arg\left(1 + \frac{g(z)}{f(z)}\right)$ but $\left|\frac{g(z)}{f(z)}\right| < 1$ so $1 + \frac{g(z)}{f(z)}$

stays inside disc $|w - 1| < 1$



so winding number $= 0$

so $\Delta_C \arg f(z) = \Delta_C \arg(f(z) + g(z))$ $\square$

Example $z^8 - 4z^5 + z^2 - 1$ (*) have inside unit circle?

$f(z) = -4z^5 \quad g(z) = z^8 + z^2 - 1. \quad |f(z)| > |g(z)|$ on $|z| = 1$.

5 zeros inside $|z|=1$ $\Rightarrow$ (*) has 5 zeros inside $|z|=1$!

Thm (Fundamental thm of algebra). Every polynomial $P(z) = a_0 + a_1 z + \dots + a_n z^n$, $a_n \neq 0$, has precisely n zeros ($n \geq 1$).

Proof set $f(z) = a_n z^n \quad g(z) = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$

consider circle of radius R, $|z| = R$: $\quad |f(z)| = a_n R^n$

$$|g(z)| \leq |a_0| + |a_1|R + \dots + |a_{n-1}|R^{n-1}$$

$$\lim_{z\to\infty} \left|\frac{f(z)}{g(z)}\right| \geq \lim_{z\to\infty} \frac{|a_n|R^n}{|a_0| + |a_1|R + \dots + |a_{n-1}|R^{n-1}} = \infty,$$

so $|f(z)| > |g(z)|$ for all R sufficiently large.

so $P(z)$ has same number of zeros as $a_n z^n$, i.e. n. $\square$.

Thm If $f(z)$ is univalent in a domain G, then $f'(z) \neq 0$ in G.

Proof let $z_0 \in G$. suppose $f'(z_0) = 0$, then $f(z) = c_0 + c_k (z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \dots$, $c_k \neq 0$, $k \geq 2$.

in some small disc $|z - z_0| < r$ inside G.

can choose disc small enough s.t. $f'(z)$ does not vanish on $0 < |z - z_0| \leq r$.

let $g(z) = c_k + c_{k+1}(z-z_0)^{k+1} + \dots$ does not vanish if $0 < |z - z_0| \leq r$

and $g(z_0) = c_k$. Let $\mu = \min_{|z-z_0|=r} \left| c_k (z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \dots \right|$

and let $a \neq 0$, $a \in \mathbb{C}$ be have $|a| < \mu$. Then $f(z) - (c_0 + a)$

$= -a + c_k (z-z_0)^k + c_{k+1}(z-z_0)^{k+1}$ has same number of zeros as

$c_k (z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \dots = (z-z_0)^k \left(c_k + c_{k+1}(z-z_0) + \dots \right)$

i.e. k zeros, and each zero is simple as $(f(z) - (c_0 + a))' = f'(z) \neq 0$.

but then $f(z) = c_0 + a$ at $k \geq 2$ distinct points. # univalent. $\square$.