bad behavior at essential singular points:

Thm (Casorati-Weierstrass) If z_0 is an essential sing pt of $f(z)$, then for any $w \in \mathbb{C} \cup \{\infty\}$, there is a sequence of pts $z_n \to z_0$ s.t. $f(z_n) \to w$.

Proof First let $A = \infty$. Suppose $f(z)$ is bounded in a deleted nbd of z_0, i.e. $|f(z)| \le M$ for all z in $0 < |z - z_0| < R$. Cauchy inequality implies $|c_n| \le \frac{M}{\rho^n}$ for all $0<\rho<R$. Let $\rho \to 0$ for $n<0$

$\Rightarrow$ $c_n = 0$ for $n<0$. $\Rightarrow$ Laurent series has no negative powers #.

so $\exists$ sequence $z_n \to z_0$ with $|f(z_n)| > n$.

Now pick $A \in \mathbb{C}$. If every deleted nbd contains z s.t. $f(z) = A$, done. spose not, then there is a deleted nbd $K = 0 < |z - z_0| < R$ s.t. $f(z) \ne A$ in ~~here~~ K.

then $g(z) = \frac{1}{f(z) - A}$ is analytic in K, and has an essential sing pt at z_0.

but then $\exists$ sequence $z_n \to z_0$ s.t. $g(z_n) \to \infty \Rightarrow f(z_n) \to A$. □.

Example $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ has an essential singularity at $z=0$.

$A = \infty$: choose $z_n = \frac{1}{n}$ then $f(z_n) = e^n \to \infty$. as $n \to \infty$.

$A = 0$: choose $z_n = -\frac{1}{n}$ then $f(z_n) = e^{-n} \to 0$ as $n \to \infty$.

$A \ne 0, \infty$: $e^{1/z} = A \Rightarrow z = \frac{1}{\ln A} = \frac{1}{(\ln A)_0 + 2\pi i n} = z_n$ so $f(z_n) = A$ and $|z_n| \to 0$. □.

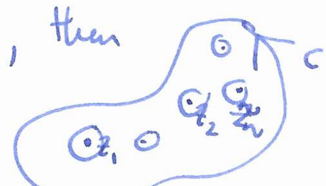
§11.3 Residues

Defn Let z_0 be an isolated sing pt of $f(z)$. The residue of f at z_0

$$\operatorname*{Res}_{z=z_0}(f(z)) = \text{coeff } c_{-1} \text{ in Laurent expansion for } f \text{ at } z_0.$$

Thm (Residue theorem) If $f(z)$ is analytic inside and on a piecewise smooth Jordan curve C, except for isolated sing pts $z_1, \dots, z_N$ inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{n=1}^{N} \operatorname*{Res}_{z=z_n} f(z)$$



<u>Proof</u> Let $\gamma_1, \dots, \gamma_N$ be small circles around the $z_1, \dots, z_N$, inside C and all disjoint. Then $\int_C f(z)\,dz = \sum_{k=1}^{N} \int_{\gamma_k} f(z)\,dz$. Let Laurent expansion of $f(z)$ at z_k be $f(z) = \sum_{k=-\infty}^{\infty} c_n^k (z-z_k)^n$ $\quad k = 1, \dots, N$.

integrate term by term: (uniform convergence on γ_k)

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} \sum_{n=-\infty}^{\infty} c_n^k (z-z_k)^n = \sum_{k=-\infty}^{\infty} c_n^k \int_{\gamma_k} \cancel{c_n^k} (z-z_k)^n\,dz$$

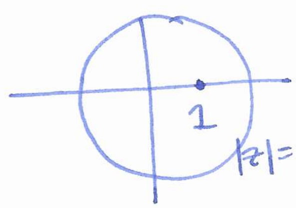
$$= \sum_{n=-\infty}^{\infty} c_n^k \int_0^{2\pi} (r_k e^{i\theta})\, d(r_k e^{i\theta}) = \sum_{k=-\infty}^{\infty} i r_k^{n+1} \underbrace{\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta}$$

$= 0$ if $n \neq -1$
$= 2\pi$ if $n = -1$.

so $\int_{\gamma_k} f(z)\,dz = 2\pi i\, c_n^k = 2\pi i \operatorname{Res}_{z_k}(f)$

$\therefore \int_C f(z)\,dz = 2\pi i \sum \operatorname{Res}_{z_k}(f)$ $\square$.

<u>Example</u> $\int_{|z|=2} \frac{e^z}{(z-1)^n}\,dz$



pole of order n at $z = 1$.

so $\int_\gamma \frac{e^z}{(z-1)^n}\,dz = 2\pi i \operatorname{Res}_{z=1} \frac{e^z}{(z-1)^n}$ Find Laurent series at $z=1$:

$$\frac{e^z}{(z-1)^n} = e \cdot \frac{e^{z-1}}{(z-1)^n} = \frac{e}{(z-1)^n}\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \dots\right.$$

$$= e\left(\frac{1}{(z-1)^n} + \frac{1}{(z-1)^{n-1}} + \frac{1}{2!\,(z-1)^{n-2}} + \dots + \frac{1}{(n-1)!\,(z-1)} + \frac{1}{n!} + \frac{z}{(n+1)!} + \dots\right)$$

so $\operatorname{Res}_{z=1} \frac{e^z}{(z-1)^n} = \frac{e}{(n-1)!}$ $\quad$ so $\int_\gamma \frac{e^z}{(z-1)^n}\,dz = \frac{2\pi i e}{(n-1)!}$

<u>Simple poles</u>: Let z_0 be a simple pole of $f(z)$, so $f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots$

then $(z-z_0)f(z) = c_{-1} + c_0(z-z_0) + c_1(z-z_0)^2 + \cdots$

$\uparrow$ ordinary power series so cts at z_0.

so $$c_{-1} = \operatorname*{Res}_{z_0} f(z) = \lim_{z\to z_0} (z-z_0)f(z).$$

<u>Special case</u>: $f(z) = \frac{g(z)}{h(z)}$ $\quad g(z_0)\neq 0$

$h(z)$ has a simple zero at z_0, i.e. $h(z_0)=0$

$h'(z_0)\neq 0$.

Then $$\operatorname*{Res}_{z_0} f(z) = \operatorname*{Res}_{z_0} \frac{g(z)}{h(z)} = \lim_{z\to z_0} \frac{(z-z_0)\,g(z)}{h(z)} = \lim_{z\to z_0} \frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}} = \frac{g(z)}{h'(z)}.$$

<u>Pole of order $m>1$</u> Laurent expansion at z_0 is $f(z) = \frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1 z + \cdots$

~~differentiate~~ so $(z-z_0)^m f(z) = c_{-m} + c_{-m+1} z + \cdots + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m + \cdots$

differentiate $(m-1)$ times: $$\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) = (m-1)!\,c_{-1} + \frac{m!}{1!}c_0(z-z_0) + \frac{(m+1)!}{2!}c_1(z-z_0)^2 + \cdots$$

so $$(m-1)!\,c_{-1} = \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$$

so $$c_{-1} = \operatorname*{Res}_{z_0} f(z) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right).$$

<u>Example</u> $f(z) = \frac{e^z}{(z-1)^n}$ $\leftarrow$ pole of order n.

$$\operatorname*{Res}_{z=1} \frac{e^z}{(z-1)^n} = \lim_{z\to 1} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^n}\left(e^z\right) = \lim_{z\to 1}\frac{e^z}{(n-1)!} = \frac{e}{(n-1)!}$$

§12 Residues

Logarithmic residues and arguments.

<u>Defn</u> The logarithmic residue of $f(z)$ at a is $\operatorname*{Res}_a \frac{f'(z)}{f(z)}$, i.e.