now integrate term by term.

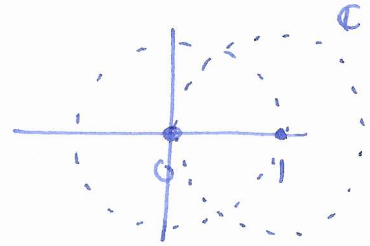
$$f(w) = \frac{1}{2\pi i}\int_{\Gamma} f(z)\sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}\,dz + \frac{1}{2\pi i}\int_{\gamma} f(z)\sum_{n=1}^{\infty}\frac{(w-a)^{-n}}{(z-a)^{-n+1}}\,dz$$

so $$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n + \sum_{n=1}^{\infty} c_{-n}(w-a)^{-n}$$

where $$c_n = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{(z-a)^{n+1}}\,dz \quad n \geq 0 \qquad c_{-n} = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^{-n+1}}\,dz \quad n \geq 1 \qquad \square.$$

Suppose $|f(z)| \leq M$ for all z in $r < |z-a| < R$.

then Cauchy inequality gives $|c_n| \leq \frac{M}{2\pi}\frac{2\pi\rho}{\rho^{n+1}} = \frac{M}{\rho^{n+1}}$ for any $r < \rho < R$.



Example $f(z) = \dfrac{1}{z(1-z)}$

in $0 < |z| < 1$ $\quad f(z) = \dfrac{1}{z} + \dfrac{1}{1-z}$.

so Laurent expansion is $\frac{1}{z} + 1 + z + z^2 + \cdots = \frac{1}{z} + \sum_{n=0}^{\infty} z^n$

in $0 < |z-1| < 1$ Laurent expansion is : $f(z) = \dfrac{1}{1+(z-1)} - \dfrac{1}{z-1}$

$$= -\frac{1}{z-1} + 1 - (z-1) + (z-1)^2 - \cdots = -\frac{1}{z-1} + \sum_{n=0}^{\infty}(-1)^n (z-1)^n.$$

Isolated singular points

Let $f(z)$ be analytic in a neighborhood of z_0, but not defined at z_0.

Then z_0 is an isolated singular point for f.

We have shown $f(z)$ has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ (*)

in some deleted neighborhood $0 < |z - z_0| < R$. There are 3 possibilities:

1) (*) contains no negative powers, we say z_0 is a removable singularity.

2) (*) contains only finitely many negative powers, we call z_0 a pole

3) (*) contains infinitely many negative powers, we call this an essential singularity

Removable singularities

full neighborhood

$f(z) =$ power series analytic in $|z-z_0|<R$, just set $f(z_0)=c_0$, get analytic function.

$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + \dots$

Poles z_0 pole, i.e. Laurent series at z_0 contains only a finite number of negative powers. $\sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$, so $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n + \frac{c_{-1}}{z-z_0} + \frac{c_{-2}}{(z-z_0)^2} + \dots + \frac{c_{-m}}{(z-z_0)^m}$

$c_{-m} \neq 0$. z_0 is a pole of order m. $m=1 \leftarrow$ simple pole
$m \geq 2 \leftarrow$ multiple pole.

note: $(z-z_0)^m f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^{n+m} + c_{-1}(z-z_0)^{m-1} + \dots + c_{-m}$ } ordinary power series

so z_0 is a removable singularity for $(z-z_0)^m f(z)$, and $\lim_{z\to z_0} (z-z_0)^m f(z) = c_{-m} \neq 0$

$\Rightarrow \lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{c_{-m}}{(z-z_0)^m} = \infty$.

Thm Let z_0 be a zero of order m for a function $f(z)$ analytic at z_0. Then $1/f(z)$ is analytic in a deleted neighborhood of z_0 and has a pole of order m at z_0.

Proof $f(z)$ has a power series of the form $f(z) = c_m (z-z_0)^m + c_{m+1}(z-z_0)^{m+1} + \dots$ $c_m \neq 0$

equivalently $f(z) = (z-z_0)^m g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$.

so $\frac{1}{f(z)} = \frac{1}{(z-z_0)^m} \frac{1}{g(z)} = \frac{h(z)}{(z-z_0)^m}$, when $h(z)$ is analytic and non-zero at z_0.

so $\frac{1}{f(z)}$ is analytic in a deleted nbd of z_0, i.e. $1/f(z)$ has a pole of order m $\square$.

Thm Let z_0 be a pole of order m of a function $f(z)$ analytic in a deleted nbd of z_0. Then $1/f(z_0)$ is analytic at z_0, provided $1/f(z_0)=0$ with a zero of order m at z_0.

Proof $\square$.

Corollary $f(z)$ non-vanishing in a deleted nbd of z_0, and has an essential sing point at z_0, then $1/f(z)$ also has an essential singular point.

Proof: If $1/f(z)$ is not essential sing pt, then it is removable, or pole of order m $\Rightarrow 1/f(z)$ has a zero $\#\square$.