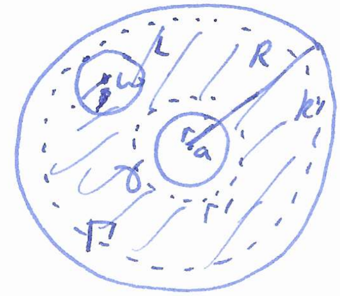
Proof series uniformly convergent on C, so so is

$$\frac{1}{2\pi i}\,\frac{f(z)}{(z-a)^{n+1}} = \sum_{k=-\infty}^{\infty} \frac{(z-a)^k}{(z-a)^{n+1}} \qquad \text{note: } \left|\frac{1}{2\pi i}\,\frac{1}{(z-a)^{n+1}}\right| = \frac{1}{2\pi\rho^{n+1}}$$

so can integrate term by term: $\displaystyle\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_C (z-a)^{k-n-1}\,dz$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi i}\int_0^{2\pi} \rho^{k-n} e^{i(k-n)\theta}\,d\theta = c_n \quad \text{as} \quad \int_0^{2\pi} e^{i(k-n)\theta}\,d\theta = 0 \text{ if } k \neq n. \qquad \square$$

Thm Let K be the annulus $r<|z-a|<R$, and suppose $f(z)$ is analytic in K. Then $f(z)$ has a Laurent series equal to $f(z)$ in K.



Proof choose $w \in K$, then $r'>r$, $R'<R$ s.t. $w \in K' = r'<|z-a|<R'$, and L small circle around w contained in K'.

$\dfrac{f(z)}{z-w}$ is analytic inside Γ, but outside γ, L.

so $$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz + \frac{1}{2\pi i}\int_L \frac{f(z)}{z-w}\,dz$$

but by Cauchy's integral formula $\displaystyle\frac{1}{2\pi i}\int_L \frac{f(z)}{z-w}\,dz = f(w)$.

so $$f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz.$$

$$= \underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz}_{} + \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{w-z}\,dz}_{\circledast}. \quad \leftarrow \text{want to write these as power series.}$$

recall: $$\frac{1}{z-w} = \cancel{\frac{1}{(z-a)\frac{z-w}{z-a}}} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)} = \frac{1}{z-a}\left(1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\dots\right)$$

$$= \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} \quad \text{here } \left|\frac{w-a}{z-a}\right| \le \frac{\rho}{R'} < 1 \text{ so converges.}$$

$\circledast$: $$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)} = \frac{1}{(w-a)}\left(1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\dots\right) = \sum_{n=1}^{\infty} \frac{(w-a)^{-n}}{(z-a)^{-n+1}}$$

here $\left|\dfrac{z-a}{w-a}\right| = \dfrac{r'}{\rho} < 1$ so converges.