where $M' = \max_{z \in \gamma_R} |f(z)|$, but then $|f(z)| = M$ on γ_R. True for all $R \leq \epsilon$

$\Rightarrow |f(z)| =$ constant M on small disc about z_0 $\Rightarrow f(z) =$ constant. $\times\!\!\times$ $\square$.

Corollary (Minimum modulus principle) $f(z)$ analytic, non-constant, $f(z) \neq 0$ in domain G. Then $|f(z)|$ does not have a minimum in G.

Proof $g(z) = \frac{1}{f(z)}$ is analytic and non-constant in G, so $|g(z)|$ has no max

$\Rightarrow |f(z)|$ has no min. $\square$.

Corollary Let $f(z)$ be analytic in a bounded domain G and cts on ∂G. then there is a point $z_0 \in \partial G$ s.t. $|f(z_0)| = \max_{z \in \partial G \cup G} |f(z)|$.

Proof f cts on $\bar{G} \Rightarrow |f(z)|$ cts on $\bar{G}$, so has a max, can't occur in interior, so must occur on ∂G. $\square$.

Corollary Let $f(z)$ be analytic in a bounded domain G, non-constant, non-vanishing. then there is a point $z_0 \in \partial G$ s.t. $|f(z_0)| = \min_{z \in G \cup \partial G} |f(z)|$.

Proof as before $\square$.

Corollary Let f be analytic, non-constant, non-vanishing in a bounded domain G. if f is constant on ∂G then f is constant on G.

Proof There are points z_0, z_1 on ∂G s.t. $|f(z_0)|$ is min and $|f(z_1)|$ is max $|f(z)|$ over all $z \in \bar{G}$. f constant on $\partial G \Rightarrow \max|f(z)| = \min |f(z)|$

$\Rightarrow |f(z)|$ is constant on $\bar{G} \Rightarrow f$ is constant on $\bar{G}$. $\square$.

Thm If $u(x,y)$ is harmonic and not constant in domain G, then $u(x,y)$ has no max or min in G.

Proof Let f be an analytic function with u as its real part, then $g(z) = e^{f(z)}$ is analytic, non-vanishing, non-constant, so $|g(z)| = e^{u(x,y)}$ has no max or min in $G \Rightarrow u$ has no max or min in G. $\square$.

Corollary Let u be harmonic in a bounded domain G, then ∂G contains pts z_0, z_1 s.t. $u(z_0) = \min_{z \in \bar{G}} u(x,y)$, $u(z_1) = \max_{z \in \bar{G}} u(z)$ $\square$.

<u>Corollary</u> Let u be harmonic for a bounded domain G, and c/s in $\bar{G}$. If u is constant on $\bar{G}$ then u is constant on G. $\square$.

§11 Laurent series

Laurent series: power series in $\frac{1}{z}$ and z

<u>Thm</u> Given a series $c_0 + \frac{c_1}{z-a} + \frac{c_2}{(z-a)^2} + \frac{c_3}{(z-a)^3} + \dots$

Let $\ell = \limsup \sqrt[n]{|c_n|}$

Then one of the following occurs:

1) if $\ell = 0$ the series is absolutely convergent for all z in $(\mathbb{C} \cup \{\infty\}) \setminus \{a\}$.
2) if $0 < \ell < \infty$ the series is absolutely convergent for all z outside $|z-a| = \ell$ and divergent for all z inside.
3) If $\ell = \infty$ the series is divergent for all finite z.

<u>Proof</u> set $w = \frac{1}{z-a}$ get $c_0 + c_1 w + c_2 w^2 + \dots$ has radius of convergence $\frac{1}{\ell} = \frac{1}{\limsup \sqrt[n]{|c_n|}}$ $\square$.

<u>Laurent series</u> $$\sum_{n=-\infty}^{\infty} c_n (z-a)^n = \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{c_n}{(z-a)^n}}_{\text{principal part}}$$

so series converges in $r < |z-a| < R$ (assuming $r < R$)

radius of convergence $R = \frac{1}{\limsup \sqrt[n]{|c_n|}}$

$r = \limsup \sqrt[n]{|c_n|}$

and is absolutely and uniformly convergent in every closed bounded subdomain.
and is analytic in annulus $r < |z-a| < R$.

<u>Thm</u> Let C be any circle in $|z-a| = \rho$ with $r < \rho < R$. Then

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \qquad n \in \mathbb{Z}.$$