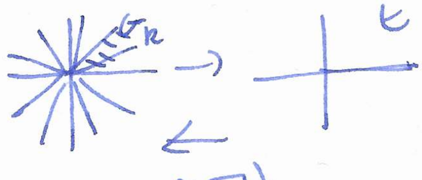
§9.3 Riemann surfaces

- $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$



$(\sqrt[n]{w})_k$.

instead of restricting the domain, we can enlarge the range.

define $S = \bigcup_{k=0}^{n-1} E_k$ ← ~~~~ n-copies of E

glued together as follow: E_k ← glue to E_{k-1}, glue to E_{k+1} (mod n).

$$\begin{tikzcd} & S \arrow[d, "n-1"] \\ \mathbb{C} \arrow[ur, "\sqrt[n]{w}", "1-1"'] \arrow[r] & \mathbb{C} \end{tikzcd}$$

- $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z$

define $S = \bigcup_{k \in \mathbb{Z}} E_k$ ← $\mathbb{Z}$ copies of E. glue as before

$$\begin{tikzcd} & S \arrow[d, "\infty-1"] \\ \mathbb{C} \arrow[ur, "z \mapsto \log(z)", "1-1"'] \arrow[r] & \mathbb{C} \end{tikzcd}$$

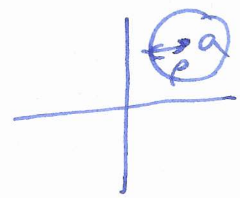
- this works for $\mathbb{C} \cup \{\infty\}$!
- warning: for different functions may need to choose different branch points/cuts.

e.g. $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2 - 2z$. inverse is $z^2 - 2z = w$, $z^2 - 2z - w = 0$, $z = \dfrac{2 \pm \sqrt{4 + 4w}}{2} = 1 \pm \sqrt{1+w}$.

so point with exactly one pre-image for f^{-1} is $w = -1$, so cut along $(-\infty, -1)$ or $(-1, \infty) \subset \mathbb{R}$.

§10 Taylor series

Let $f(z)$ be analytic in a domain containing a, so $f^{(n)}(a)$ all exist. then $c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$ where $c_n = \dfrac{f^{(n)}(a)}{n!}$ is the Taylor series for f based/centered at a.



suppose Taylor series has radius of convergence R and pick $\rho < R$.

Cauchy's integral formula: $f(a) = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{z-a}\, dz$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

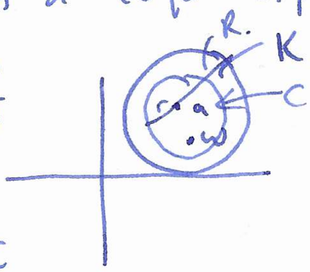
so $c_n = \dfrac{1}{2\pi i} \displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$, so if $|f(z)| \le M$ for all $|z| < R$ then

$$|c_n| \le \frac{M}{2\pi} \frac{2\pi\rho}{\rho^{n+1}} = \frac{M}{\rho^n}$$

take limit as $\rho \to R \Rightarrow |c_n| \le \frac{M}{R^n}$

Thm Let K be the disc $|z-a|<R$, $f(z)$ analytic in K. Then $f(z)$ has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$.

Proof



$w \in K$. $f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}dw \leftarrow$ want to write this as a power series

$z \in C$

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}$$

note $\left|\frac{w-a}{z-a}\right| < 1$ by suitable choice of C (can assume)

$$= \frac{1}{(z-a)}\left(1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \dots\right) = \sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}$$

so $\frac{1}{2\pi i}\frac{f(z)}{z-w} = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{f(z)(w-a)^n}{(z-a)^{n+1}}$

set $M = \max_{z\in C}|f(z)|$

then $\left|\frac{1}{2\pi i}\frac{f(z)(w-a)^n}{(z-a)^{n+1}}\right| \le \frac{M}{2\pi r}\underbrace{\left(\frac{w-a}{r}\right)^n}_{<1}$.

so sum is uniformly convergent on C
now integrate term by term.

so $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}dw = \sum_{n=0}^{\infty} c_n(w-a)^n$, when $w_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{f^{(n)}(a)}{n!}$ $\square$.

Defn If $f(z)$ is analytic in a nbd of z, z is a regular pt, otherwise z is a singular point.

Example $f(z) = \frac{1}{z}$ 0 singular, all other points regular.

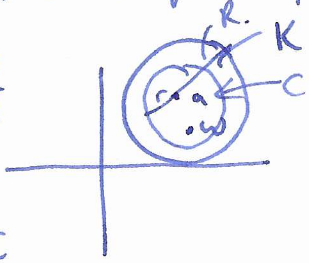
Thm Suppose $f(z)$ has Taylor expansion $f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$ with radius of convergence R. Then all points in $|a-z|<R$ are regular, and there is at least one singular point on the circle $|a-z|=R$.

Proof spse every point on $C = \{|a-z|=R\}$ regular, then each one contained in a disc D_z in which $f(z)$ analytic, [HB] $\Rightarrow$ finitely many discs cover C $D_{z_1} \dots D_{z_k}$ but then $f(z)$ analytic in $|a-z|<R+\epsilon$ for ϵ = min radius D_{z_i}, but then $f(z)$ has convergent Taylor series in $|a-z|<R+\epsilon$, which agrees w/ series in $|a-z|<R$, so they must be the same, so radius of convergence $>R$ $\lightning$ $\square$.

take limit as $\rho \to R$ $\Rightarrow$ $|c_n| \le \frac{M}{R^n}$

Thm Let K be the disc $|z-a| < R$, $f(z)$ analytic in K. Then $f(z)$ has a Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$.

Proof



$w \in K$. $f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dw$ ← want to write this as a power series

$z \in C$

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}$$

note $\left|\frac{w-a}{z-a}\right| < 1$ by suitable choice of C (can assume)

$$= \frac{1}{(z-a)}\left(1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \dots\right) = \sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}$$

so $\frac{1}{2\pi i}\frac{f(z)}{z-w} = \frac{1}{2\pi i}\sum_{n=0}^{\infty} f(z)\frac{(w-a)^n}{(z-a)^{n+1}}$

set $M = \max_{z\in C}|f(z)|$

then $\left|\frac{1}{2\pi i} f(z)\frac{(w-a)^n}{(z-a)^{n+1}}\right| \le \frac{M}{2\pi r}\underbrace{\left(\frac{|w-a|}{r}\right)^n}_{<1}$.

so sum is uniformly convergent on C
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so $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dw = \sum_{n=0}^{\infty} c_n (w-a)^n$, where $c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{f^{(n)}(a)}{n!}$ □.

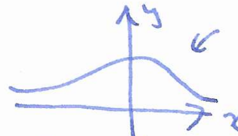
Defn If $f(z)$ is analytic in a nbd of z, z is a regular pt, otherwise z is a singular point.

Example $f(z) = \frac{1}{z}$ 0 singular, all other points regular.

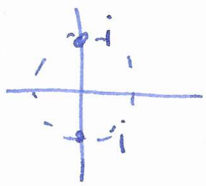
Thm Suppose $f(z)$ has Taylor expansion $f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots$ with radius of convergence R. Then all points in $|a-z| < R$ are regular, and there is at least one singular point on the circle $|a-z| = R$.

Proof suppose every point on $C = \{|a-z| = R\}$ regular, then each one contained in a disc D_z in which $f(z)$ analytic, [HB] $\Rightarrow$ finitely many discs cover C $D_{z_1} \dots D_{z_k}$, but then $f(z)$ analytic in $|a-z| < R+\epsilon$ for ϵ = min radius D_{z_i}, but then $f(z)$ has convergent Taylor series in $|a-z| < R+\epsilon$, which agrees w/ series in $|a-z| < R$, so they must be the same, so radius of convergence $> R$ ↯ □.

Example $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$



← non-sing. as real function.



$\mathbb{C}$. but sing points at $\pm i$ so radius of convergence $= 1$.

Thm (Liouville) every bounded entire function is constant.
i.e. if $f : \mathbb{C} \to \mathbb{C}$ defined on all of $\mathbb{C}$, analytic, ~~then~~ and $|f(z)| \le M$ $\forall z \in \mathbb{C}$, then $f(z) = c$ for some $c \in \mathbb{C}$.

Proof. $f(z)$ has a Taylor expansion $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ for all $z \in \mathbb{C}$. ($R = \infty$)
in particular for $|z| < R$.

Cauchy's inequality: $|c_n| \le \dfrac{M}{R^n} \to 0$ as $R \to \infty$ (hold for $n \ge 1$). $\square$.

§10.2 Uniqueness

we have shown if two power series $c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$ and $d_0 + d_1(z-a) + c_2(z-a)^2 + \cdots$ (same a!) define same function then $c_n = d_n$ for all n.

don't need them to coincide on an open disc:

Thm (uniqueness for power series) If the sum of two power series in $(z-a)$ coincide at every point of a set E with a as a limit pt of E, then they have the same coefficients.

Proof let z_n be a sequence in E converging to a. $\sum_k a_k (z_n - a)^k = \sum_k b_k (z_n - a)^k$
Sum of power series cts inside radius of convergence, for all $z_n \in E$.
so $a_0 = \lim_{n \to \infty} \sum a_k (z_n - a)^k = \lim_{n \to \infty} \sum b_k (z_n - a)^k = b_0$ so $a_0 = b_0$.

now do induction on k: $a_1 (z_n - a) + a_2 (z_n - a)^2 + \cdots = b_1 (z_n - a) + b_2 (z_n - a)^2 + \cdots$

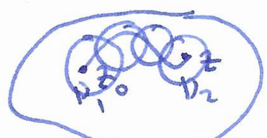
divide by $(z_n - a)$: $a_1 + a_2 (z_n - a) + \cdots = b_1 + b_2 (z_n - a)^2 + \cdots$

so above shows $a_1 = b_1$, etc. $\square$.

Thm (Uniqueness for analytic functions) Let f, g be two functions analytic in G, equal at $E \subseteq G$, and E has limit pt $z_0 \in G$. Then $f = g$ in G.

Remark: if G is a disc, we are done by above.

Proof



given $z \in G$, choose a curve C from z_0 to z.
[BW] can cover C by finitely many discs $D_1, \ldots D_n$.
D_1 contains limit pt z_0, so $f = g$ on $D_1 \Rightarrow f = g$ on $D_1 \cap C$
but $D_1 \cap C$ is an arc, which contains a limit point in D_2, so $f = g$ on D_2, etc. $\square$

Note if $f(z) = \overset{c}{\text{const}}$ on any set with a limit pt in G, then $f = c$ on G!

Let $f(z)$ be analytic in G, and let z_0 be a zero of f, i.e. $f(z_0) = 0$.
(to function)

so f has a power series expansion at z_0: $\underset{\overset{\shortparallel}{0}!}{c_0} + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$

Note: at least one c_i must be non-zero, otherwise $f = 0$ on an open set containing $z_0 \Rightarrow f = 0$. So $f(z) = c_m(z-z_0)^m + c_{m+1}(z-z_0)^{m+1} + \cdots$ c_m first non-zero coefficient.

then we say <u>z_0 is a zero of order m</u>. z_0 is a <u>simple zero</u> if $m = 1$ else <u>multiple zero</u>.

<u>Thm</u> Let z_0 be a zero of f analytic in G. Then there is a nbd of z_0 s.t. f has no other zeros in the nbd.

<u>Proof</u> suppose not, then every nbd $B(z_0, \frac{1}{n})$ contains a zero $z_n \neq z_0$
$\Rightarrow z_0$ is a limit point of zeros, $\Rightarrow f \equiv 0$. $\square$.

<u>Thm</u> Let f be analytic in G. If $|f(z)|$ is constant in G, then f is constant in G.

<u>Proof</u> If $|f(z)| = 0$ in G, then $f(z) = 0$ in G. Suppose $|f(z)| = M > 0$ in G. Let $f = u + iv$,
$|f(z)|^2 = u^2 + v^2 = M^2$ in G. Differentiate:

$$\left.\begin{aligned} 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} &= 0 \\ 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \text{ in } G$$

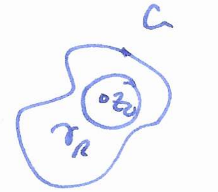
u, v cannot both be zero in G as $M > 0$.

$$\Rightarrow \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0 \quad \text{use CR equations: } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0 \quad \Rightarrow f = \text{const}. \ \square.$$

§ 10.3 Maximum modulus principle

<u>Thm</u> f analytic, non-constant in G, then $|f(z)|$ does not have a maximum in G.



<u>Proof</u> Suppose there is $z_0 \in G$ s.t. $\underset{M}{\overset{}{|f(z_0)|}} \geq |f(z)|$ for all $z \in G$.

Let γ_R be a small circle centred at z_0 contained in G.

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{Re^{i\theta}}\, iRe^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta$$

so $|f(z_0)| = M = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + Re^{i\theta})|\,d\theta \leq \frac{1}{2\pi}2\pi M$!