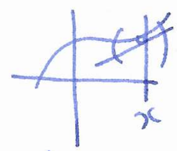
• E is open. Let $w \in E$, and $z \in G$ s.t. $f(z) = w$.

apply implicit function theorem: 1d case:



$y = f(x)$ if $y'(x) \neq 0$ and x 1-1 then locally near x $x = g(y)$.

2d case: • need f 1-1.

• need all partial derivatives exist and are cts

• need Jacobian $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = |f'(z)|^2 \neq 0$.

<u>Fact</u> if $f: G \to \mathbb{C}$ is 1-1 then $f'(z) \neq 0$.

implicit function guarantees w has open nbd contained in E $\square$.

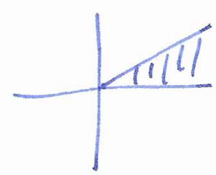
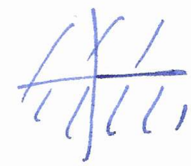
<u>Thm</u> Let $f: G \to E$ be analytic bijection, and let f^{-1} be the inverse function. Then f^{-1} is univalent, and $(f^{-1}(w))' = \dfrac{1}{f'(z)}$.

<u>Proof</u> • f^{-1} exists as f 1-1 and onto.

• f^{-1} cts by implicit value theorem. Let $w_0 = f(z_0)$. Then $z \to z_0$ iff $f(z) \to f(z_0)$, $w \to w_0$.

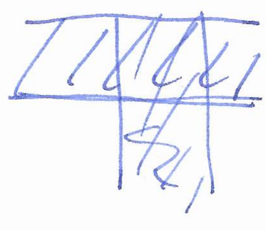
• $f^{-1\prime}(w_0) = \lim_{w \to w_0} \dfrac{z - z_0}{w - w_0} = \lim_{z \to z_0} \dfrac{1}{\frac{w - w_0}{z - z_0}} = \dfrac{1}{f'(z)}$. $\square$.

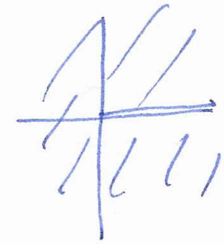
<u>Examples</u> ① $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ univalent in $0 < \arg(z) < \frac{2\pi}{n}$.


$\to$


$f^{-1}: \mathbb{C} \to \mathbb{C}$, $z \mapsto \sqrt[n]{z}$ $\quad \dfrac{d}{dw}(\sqrt[n]{w}) = \dfrac{1}{\frac{dz^n}{dz}} = \dfrac{1}{nz^{n-1}} = \dfrac{\sqrt[n]{w}}{nw} = \dfrac{1}{n} w^{\frac{1}{n} - 1}$

② $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z$ univalent in $0 < \operatorname{Im} z < 2\pi$





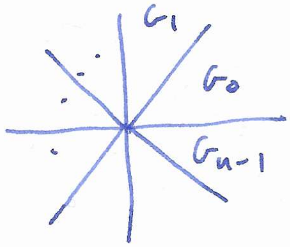
$|f(z)| = e^x \quad \arg(f(z)) = y$

$f^{-1}: \mathbb{C} \to \mathbb{C}$, $z \mapsto \log(z)$ $\quad \dfrac{d}{dw}(\log w) = \dfrac{1}{\frac{de^z}{dz}} = \dfrac{1}{e^z} = \dfrac{1}{w}$.

$\to$ so $\log(w) = \ln|w| + i\arg(w)$.

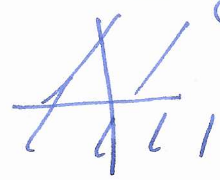
§9.2 Branches and branch points

- consider $f:\mathbb{C}\to\mathbb{C}$, $z\mapsto z^n$. consider wedges $\frac{2k\pi}{n} < \arg z < \frac{2(k+1)\pi}{n}$ for $k=0,\dots,n-1$



$n=8$

each G_i gets mapped to

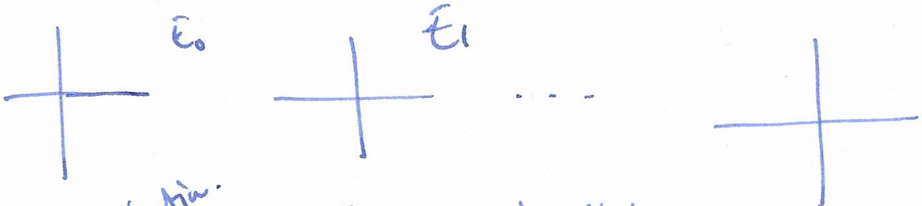


$E = \mathbb{C}\setminus\mathbb{R}_+$

$f|_{G_k}: G_k \to E$ is bijection so has inverse. let inverse be $(\sqrt[n]{w})_k$

so we can define a (multiple valued) inverse $f^{-1}: E \to \mathbb{C}^n$, $w \mapsto ((\sqrt[n]{w})_0, \dots (\sqrt[n]{w})_n)$

alternative construction: take n copies of E, $E_0, \dots E_{n-1}$

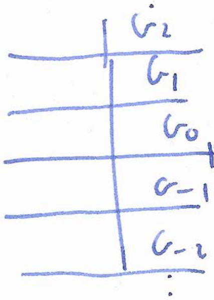


and glue them together as shown

E_k glue to E_{k-1}, glue to E_{k+1}.

gives
$$\begin{tikzcd} & \sqcup E_k \arrow[d, "n-1"] \\ \mathbb{C} \arrow[ur, "\text{bijection}"] \arrow[r, "n-1"'] & \mathbb{C} \end{tikzcd}$$
← can show this doesn't depend on choices!

- consider $f:\mathbb{C}\to\mathbb{C}$, $z\mapsto e^z$. divide domain into strips $2k\pi < \operatorname{Im} z < 2(k+1)\pi$

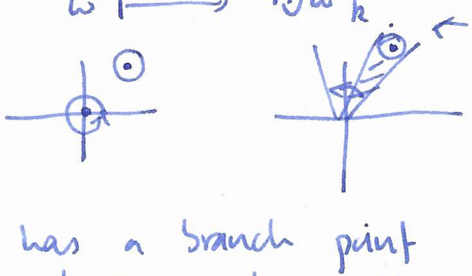


E as above: $f|_{G_k}: G_k \to E$ is a bijection.

so has inverse $f_k^{-1} = \ln(w)_k : E \to G_k$.

can define multiple valued $\log: \mathbb{C} \to \mathbb{C}^{\mathbb{N}}$, $w \mapsto (\log(w)_k)_k$.

branch points: $E \to \mathbb{C}$, $w \mapsto \sqrt[n]{w}_k$



← regular point.

0 is a branch point as if small loops around 0 change the branch

$w \mapsto \sqrt[n]{w}$ has a branch point of order n at 0, and no other branch points.

$w \mapsto \log(w)$ has a branch point of infinite order at 0