③ $f(z) = z+b$ is circle preserving. (translation)

Finally: $f(z) = \dfrac{az+b}{cz+d}$ is a composition of maps of the form $z \mapsto az$, $z \mapsto \frac{1}{z}$, $z \mapsto z+b$.

do long division:

$$\begin{array}{r} a/c \\ cz+d \,\big)\, \overline{az+b} \\ az+\frac{ad}{c} \\ \hline b-\frac{ad}{c} \end{array}$$

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cz+d}. \checkmark. \quad \square.$$

Thm Given three ordered distinct points a,b,c there is a unique* Möbius map taking them to $0,1,\infty$.

Proof existence:
$a \mapsto 0$
$b \mapsto 1$
$c \mapsto \infty$

$$\frac{(b-c)(z-a)}{(b-a)\,z-c} \quad \cancel{\frac{bz-ab}{z-c}}.$$

uniqueness: if $a \mapsto 0$ numerator must factor as $*(z-a)$
if $c \mapsto \infty$ denominator must factor as $(z-c)$.
so f defined up to ratio, but fixed by $b \mapsto 1$. $\square$.

*: Möbius transformation not uniquely represented by a,b,c,d.

$\dfrac{az+b}{cz+d}$ same as $\dfrac{2az+2b}{2cz+2d}$.

Defn cross ratio of x,y,z,t is $\dfrac{z-x}{z-y}\,\dfrac{t-x}{t-y}$

Thm cross ratio is preserved by Möbius transformations.

Proof check: $z \mapsto az$: $\dfrac{az-ax}{az-ay}\cdot\dfrac{at-ax}{at-ay} = \dfrac{z-x}{z-y}\,\dfrac{t-x}{t-y}$

$z \mapsto z+b$: $\dfrac{z+b-x-b}{z+b-y-b}\cdot\dfrac{t+b-x-b}{t+b-y-b} = \dfrac{z-x}{z-y}\,\dfrac{t-x}{t-y}$

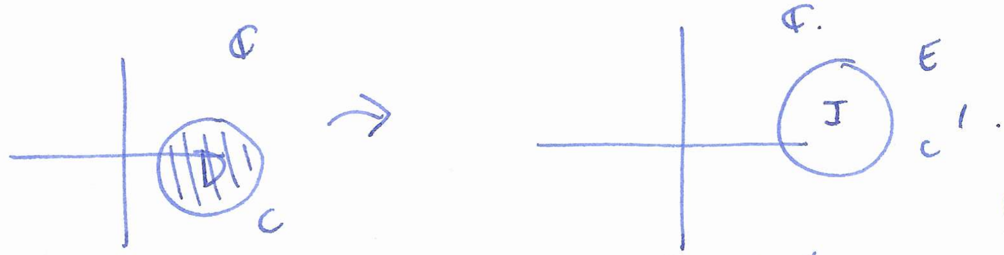
$z \mapsto \frac{1}{z}$: $\dfrac{\frac{1}{z}-\frac{1}{x}}{\frac{1}{z}-\frac{1}{y}}\cdot\dfrac{\frac{1}{t}-\frac{1}{x}}{\frac{1}{t}-\frac{1}{y}} = \dfrac{\frac{x-z}{xz}}{\frac{y-z}{zy}}\cdot\dfrac{\frac{x-t}{xt}}{\frac{y-t}{ty}} = \dfrac{x-z}{y-z}\,\dfrac{x-t}{y-t} = \dfrac{z-x}{z-y}\cdot\dfrac{x-t}{\cancel{y}\,t-y} \quad \square$

Geometric interpretation(s).

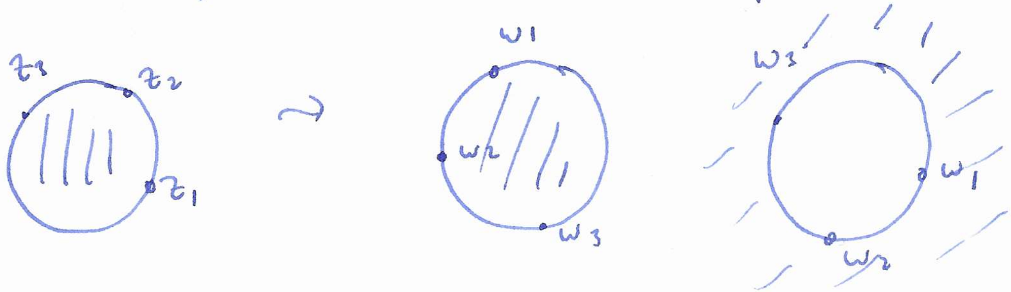
$[x,y,z,t] \mapsto \left(0,1,\infty,\dfrac{y-z}{y-x}\cdot\dfrac{t-x}{t-z}\right) \quad \left(0,\cancel{1}^{\,\infty},\cancel{\infty}^{\,1},\dfrac{z-x}{z-y}\cdot\dfrac{t-x}{t-y}\right)$

Corollary Let C be circle determined by $z_1, z_2, z_3 \in \mathbb{C}$.
C' w_1, w_2, w_3.
then there is a unique Möbius map taking C to C' s.t. $z_1 z_2 z_3$ go to w_1, w_2, w_3. $\square$

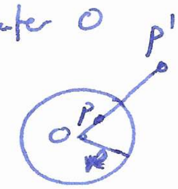
Remark.



D can go to either I or E. -which one?

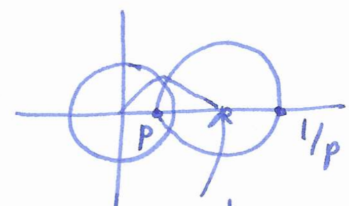


Recall P, P' are symmetric wrt circle C, if lie on same ray through center O and $OP \cdot OP' = R^2$



Thm Two points P, P' are symmetric wrt C iff every circle or straight line through P and P' is orthogonal to C.

Proof $\Rightarrow$. wlog, center $O = 0 \in \mathbb{C}$, $R = 1$, P, P' lie on the x-axis.



$\frac{p+\frac{1}{p}}{2}$

$$\frac{p+\frac{1}{p}}{2} - p = \frac{\frac{1}{p} - p}{2}$$

$\frac{p+\frac{1}{p}}{2}$

check: $$1^2 + \left(\frac{\frac{1}{p} - p}{2}\right)^2 = 1 + \frac{\frac{1}{p^2} - 2 + p^2}{4} = \frac{\frac{1}{p^2} + 2 + p^2}{4} = \left(\frac{\frac{1}{p} + p}{2}\right)^2$$

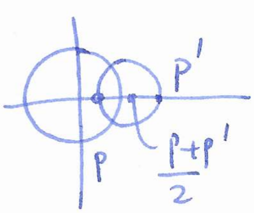
so right angled triangle $\Rightarrow$ tangents $\perp$ $\boxtimes$

$\Leftarrow$. suppose every circle and straight line through P, P' orthogonal to C.
$\Rightarrow$ lie on common ray through O. Now pick any circle through P, P'

circles $\perp$ $\Rightarrow$

$\frac{p+p'}{2} - p = \frac{p'-p}{2}$ right angled.

$\frac{p+p'}{2}$



$$1^2 + \left(\frac{p'-p}{2}\right)^2 = \left(\frac{p+p'}{2}\right)^2 \Rightarrow 4 + \cancel{p'^2} - 2pp' + \cancel{p^2} = \cancel{p^2} + 2pp' + \cancel{p'^2}$$
$$\Rightarrow pp' = 1. \checkmark \quad \square.$$

Thm Let z_1 and z_2 be symmetric wrt C, let f be a Möbius transformation then $f(z_1)$ and $f(z_2)$ are symmetric wrt $f(C)$.

Proof f takes circles / straight lines to circles / straight lines, preserves angles. $\square$.

Useful facts about Möbius transformations

- $f(z) = \dfrac{az+b}{cz+d}$ $\quad g(z) = \dfrac{ez+f}{hz+k}$ $\quad g(f(z)) = ?$ $\begin{bmatrix} e & f \\ h & k \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

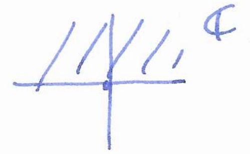
  $\updownarrow$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ $\updownarrow$ $\begin{bmatrix} e & f \\ h & k \end{bmatrix}$.

  this gives a map 2x2 matrices $\to$ Möbius maps

  $SL_2\mathbb{C}$

  Q: what is the kernel?

- what if a,b,c,d real? get $SL_2\mathbb{R} \subset SL_2\mathbb{C}$. preserves $\mathbb{R}$ in $\mathbb{C}$.



- models for $\mathbb{H}^2$: upper half space  straight lines / geodesics $\leftrightarrow$ lines / circles $\perp$ to $\mathbb{R}$ or ∂disc.

  disk 

  $SL_2\mathbb{R}$.

  $\leftrightarrow$?

- isometries? reflections, rotations, translations, parabolics.
- models for $\mathbb{H}^3$:  $\mathbb{C}\cup\{\infty\}$ $\quad \mathrm{Isom}^+\mathbb{H}^3 \leftrightarrow SL_2\mathbb{C}$.

Note: $\mathrm{Isom}(\mathbb{H}^n)$ acts transitively on $\mathbb{H}^n$, so can send any point to any other point.

- pairs of points?
- lines?

§9 Multiple valued functions

idea:

problem

$f: z \mapsto z^2$ is 2-1, so f^{-1} is not defined.

solution: define $f: \mathbb{C} \to \mathbb{C}^2$

$z \mapsto (+\sqrt{z}, -\sqrt{z})$

Defn $f: G \to \mathbb{C}$ (domain G) is **univalent** if it is a 1-1 function. **Fact** then $f'(z) \neq 0$.

Thm Let $f: G \to \mathbb{C}$ be analytic in the domain G and univalent / injective. Then $E = f(G)$ is also a domain in $\mathbb{C}$.

Proof • path connected if $w_1, w_2 \in E$ then $\exists$ $z_1, z_2 \in G$ s.t. $f(z_1) = w_1$, $f(z_2) = w_2$. As G is connected there is an arc $c: [a,b] \to G$ s.t. $c(a) = z_1$ and $c(b) = z_2$. Then $f(c(t))$ is an arc in E, so E is path connected.