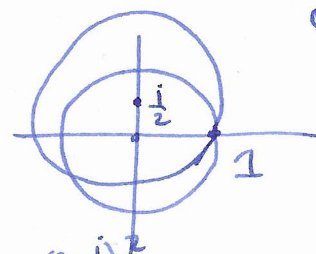
Remark if f analytic in a disc, then f ~~can be repn~~ is equal to a power series centered at the center of the disc, so analytic functions in a disc correspond 1-1 w/ power series in a disc. Which power series?

$$f(z) = f(z_0) + f'(z_0)z + \frac{f''(z_0)}{2!}z^2 + \cdots$$



Example $1+z+z^2+z^3+\cdots = \frac{1}{1-z}$ in $|z|<1$

Q: what about power series at $z = \frac{i}{2}$?

$f(z) = (1-z)^{-1}$ $\qquad f_{\frac{i}{2}}(z) = \frac{1}{1-\frac{i}{2}} + \frac{1}{(1-\frac{i}{2})^2}(z-\tfrac{i}{2}) + \frac{1}{(1-\frac{i}{2})^3}(z-\tfrac{i}{2})^2 + \cdots$

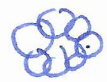
$f'(z) = (1-z)^{-2}$

$f''(z) = 2(1-z)^{-3}$. $\qquad$ radius of convergence : ratio test $\lim_{n\to\infty} \frac{c_{n+1}(z-\frac{i}{2})^{n+1}}{c_n(z-\frac{i}{2})^n}$

$f^{(n)}(z) = n!(1-z)^{-n-1}$ $\qquad \lim_{n\to\infty} \frac{(1-\frac{i}{2})^n(z-\frac{i}{2})^{n+1}}{(1-\frac{i}{2})^{n+1}(z-\frac{i}{2})^n} = \lim_{n\to\infty} \frac{z-\frac{i}{2}}{1-\frac{i}{2}}$ $\qquad$ converges for $|z-\frac{i}{2}| < |1-\frac{i}{2}|$.

$\sqrt{5}/2$.

can extend f to slightly larger domain. warning.



?

§8 Special functions

Def^n A function $f(z)$ is <u>entire</u> if it is analytic for all $z \in \mathbb{C}$.

Def^n $e^z = 1+z+\frac{z^2}{2!}+\cdots$ $\qquad \cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}\cdots$ $\qquad \sin z = z-\frac{z^3}{3!}+\cdots$

Note $R=\infty$, so these are <u>entire</u> complex functions.

Prop^n $e^{z_1+z_2} = e^{z_1}e^{z_2}$

Proof $e^{z_1} = 1+z_1+\frac{z_1^2}{2!}+\cdots$ $\qquad e^{z_2} = 1+z_2+\frac{z_2^2}{2!}+\cdots$

$$e^{z_1}e^{z_2} = \left(1+z_1+\frac{z_1^2}{2!}+\cdots\right)\left(1+z_2+\frac{z_2^2}{2!}+\cdots\right) = 1+(z_1+z_2)+\underbrace{\frac{z_1^2}{2!}+z_1z_2+\frac{z_2^2}{2!}}_{\frac{(z_1+z_2)^2}{2!}}+\cdots$$

$$= e^{z_1+z_2}$$ $\square$.

Furthermore $e^{iz} = \left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)+i\left(z-\frac{z^3}{3!}+\cdots\right) = \cos z + i\sin z$ (Euler's formula).

similarly: $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ $\qquad \sin z = \frac{e^{iz}-e^{-iz}}{2}$.

Note. e^z is periodic, with period $2\pi i$: $e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z$

- polar form $z = x + iy = re^{i\theta} = r\cos\theta + ir\sin\theta$
- $e^z \neq 0$ for all $z \in \mathbb{C}$, as $|e^z| = |e^x \cos\theta + ie^x \sin\theta| = |e^x|\,|\cos\theta + i\sin\theta|$ (where $|\cos\theta + i\sin\theta| = 1$) $= |e^x| \neq 0$.

Trig identities

- $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
- $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

follows from: $e^{i(z_1+z_2)} = e^{iz_1} e^{iz_2}$

$$\cos(z_1+z_2) + i\sin(z_1+z_2) = (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$$
$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

can check: $\cos(z+2\pi) = \cos(z)$
$\sin(z+2\pi) = \sin(z)$

set $z_1 = -z_2$, $\cos 0 = \cos^2 z_1 + \sin^2 z_1 = 1$.

warning: $|\cos z| \not\leq 1$. $\cos i = \cosh 1 > 1$ $\quad \sin i = i\sinh 1 \quad |\cdot| > 1$.

Q: where are the zeros of $\sin z$ and $\cos z$?

solve $\sin z = 0$ $\quad \dfrac{e^{iz} - e^{-iz}}{2} = 0 \quad e^{iz} = e^{-iz} \quad e^{2iz} = 1$

$$e^{2i(x+iy)} = 1 \qquad \underbrace{e^{2ix}}_{\text{arg}} \underbrace{e^{-2y}}_{\text{mod.}} = 1 \qquad e^{-2y} = 1 \Rightarrow y = 0.$$

$2ix = 2\pi i n \quad x = \pi n. \quad n \in \mathbb{Z}.$

so $\sin z = 0$ iff $z = \pi n$, $n \in \mathbb{Z}$.

Similarly: $\cos z = 0$ iff $z = \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$.

Hyperbolic functions recall $\cosh z = \frac{e^z + e^{-z}}{2}$ $\sinh z = \frac{e^z - e^{-z}}{2}$

power series: $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

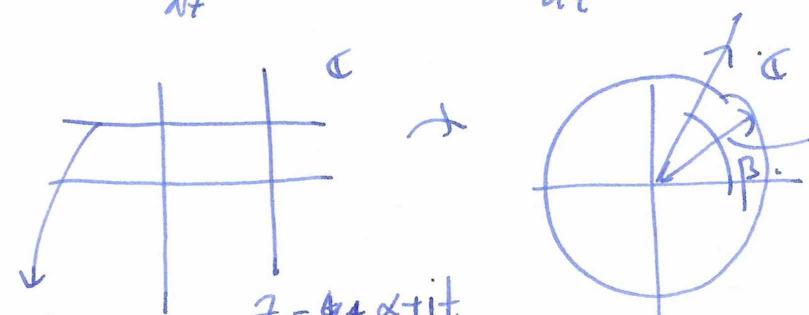
So: $\cosh z = \cos iz$ $\cos iz = \cosh z$
$\sinh z = -i \sin iz$ $\sin iz = i \sinh z$
} and similar but different trig identities.

Derivatives: can just differentiate power series.

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$ $\frac{d}{dz} e^z = 1 + z + \frac{z^2}{2!} + \dots = e^z$

$\frac{d}{dz}(\sin z) = \cos z$ $\frac{d}{dz}(\cos z) = -\sin z$ $\frac{d}{dz}(\sinh z) = \cosh z$ $\frac{d}{dz}(\cosh z) = \sinh z$.

$f: \mathbb{C} \to \mathbb{C}$
$z \mapsto e^z$



$z = t + \beta i$ $z = \alpha + it$

$\mapsto e^{t+\beta i}$ $e^t(\cos\beta + i\sin\beta)$. infinite open ray.

$\mapsto e^{\alpha + it} = e^\alpha(\cos t + i \sin t)$. circle

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Möbius maps $f(z) = \frac{az+b}{cz+d}$ $ad - bc \neq 0$.

Thm Möbius maps preserve {circles and straight lines}.

Proof recall: general equation of circle/straight line is $A|z|^2 + Bz + \bar{B}\bar{z} + D = 0$
$A, D \in \mathbb{R}$ $B \in \mathbb{C}$.

check special cases:
① $f(z) = az$ $\frac{f(z)}{a} = z$ sub in: $A\frac{|f(z)|^2}{a^2} + \frac{Bf(z)}{a} + \frac{\bar{B}\overline{f(z)}}{\bar{a}} + D = 0$

still equation of circle or straight line ✓.

② $f(z) = \frac{1}{z}$ sub $z = \frac{1}{f(z)}$: $\frac{A}{|f(z)|^2} + \frac{B}{f(z)} + \frac{\bar{B}}{\overline{f(z)}} + D = 0$

$A + B\overline{f(z)} + \bar{B}f(z) + D|f(z)|^2 = 0$ ✓ equation of circle/straight line ✓.