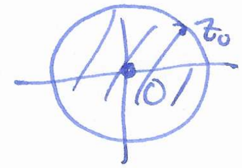
Proof spse $c_0 + c_1 z_0 + c_2 z_0^2 + \cdots$ converges. then $\lim_{n\to\infty} c_n z_0^n = 0$

so $c_n z_0^n$ bounded sequence, $|c_n z^0| \le M \ \forall n$.

if $|z| < |z_0|$ then $|c_n z^n| = |c_n z_0^n| \left|\frac{z}{z_0}\right|^n < M \left|\frac{z}{z_0}\right|^n = Mr^n$ $\quad r < 1$.

so $|c_0 + c_1 z + \cdots| \le |c_0| + |c_1 z| + \cdots < M + Mr + Mr^2 + \cdots = \frac{M}{1-r}$

so absolutely convergent. $\square$.



Example (radius of convergence $= \infty$).

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ $\quad |e^z| \le |1| + |z| + \left|\frac{z^2}{2!}\right| + \cdots$
$\le 1 + |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots$

ratio test: $\lim_{n\to\infty} \frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^n} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0 \Rightarrow$ converges.

Thm Suppose $\sum a_n z^n$ converges for at least one $z \ne 0$. Then there is a number $R > 0$ s.t. it converges for all $|z| < R$ and diverges for all $|z| > R$.

Q: what about $|z| = R$?

Proof consider $R = \sup |z|$ over all z for which series converges. if it converges for some $|z| > R$ $\cancel{\#}$ $\square$.
If $|z| < R$ then converges by above.

Example: $1 + z + z^2 + \cdots = \frac{1}{1-z}$ if $|z| < 1$ so $R = 1$.

note: not uniformly convergent on $|z| < 1$, but is on $|z| \le r < 1$.

Thm If $\sum c_n z^n$ has radius of convergence R, then it is uniformly convergent on $|z| \le r < 1$.

Proof $|c_n z^n| = |c_n||z|^n \le |c_n| r^n$ converges for all $r < R$ $\square$.

Thm $s(z) = \sum c_n z^n$ is cts in $|z| < R$. Pf as unif cts for $|z| < r < R$ $\square$.

Thm Let $s(z) = \sum c_n z^n$ have radius of convergence R. then $s(z)$ is analytic in $|z| < R$ and furthermore can be differentiated term by term so

$s^{(n)}(z) = c_1 + 2c_2 z + 3c_3 z^2 + \cdots + nc_n z^{n-1} + \cdots$ $\leftarrow$ with same radius of convergence.

Proof recall: $\sum c_n z^n$ is uniformly convergent in $|z| \le r < R$, and hence in every bounded closed domain $\bar{D}$ in $|z|<R$. each $c_n z^n$ is analytic in $|z|<R$ (in fact in $\mathbb{C}$!), so $s(z)$ is analytic in $|z|<R$, and can be differentiated term by term. Let R' = radius of convergence of $\sum c_n n z^{n-1}$, then $R \le R'$ as $c_n z^n = n c_n z^{n-1} \left(\frac{z}{n}\right)$ <1 for $n >> 0$. Spse $R' > R$, then $\sum c_n n z^{n-1}$ uniformly convergent in $R+\epsilon$, can integrate, $\Rightarrow$ $\sum c_n z^n$ converges in $R+\epsilon$ $\#$ $\square$.

Remark: can differentiate once $\Rightarrow$ can differentiate infinitely often

§7.2 Radius of convergence

Def^n (a_n) sequence of non-negative real numbers.

upper limit $= \overline{\lim\limits_{n\to\infty}}\, a_n = \limsup\limits_{n\to\infty} a_n =$ largest limit point of a_n if a_n is bounded.

$= +\infty$ otherwise..

Fact if (a_n) converges: $\lim a_n = \overline{\lim}\, a_n$ **Fact** $\overline{\lim\limits_{n\to\infty}}\, a_n$ always exists.

Fact lim inf.

Th^m (Cauchy–Hadamard) $\sum\limits_{n=0}^{\infty} c_n z^n$ power series, set $\ell = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|c_n|}$

then the radius of convergence is $R = \frac{1}{\ell}$, where $R = \begin{cases} +\infty & \text{if } \ell = 0 \\ 0 & \text{if } \ell = +\infty \end{cases}$

Proof case 1 $\ell = +\infty$, so $\sqrt[n]{|c_n|}$ unbounded want: $R=0$.

spose $\sum c_n z_0^n$ converges for some $z_0 \ne 0$, then $\lim\limits_{n\to\infty} c_n z^n = 0$, so there is $M>0$ $(M>1!)$.

s.t. $|c_n z_0^n| < M$ for all n; so $\sqrt[n]{|c_n|}\,|z_0| < \sqrt[n]{M}$, so $\sqrt[n]{|c_n|} < \sqrt[n]{M}\frac{1}{|z|}$

$\#$ sequence unbounded. $\square$.

case 2 $\ell = 0$ want: $R = +\infty$. $\ell = 0 \Rightarrow \sqrt[n]{|c_n|} \to 0$ as $n \to \infty$.

so $\sqrt[n]{|c_n|} < \epsilon$ for all $\epsilon > 0$, all n sufficiently large, so $\sqrt[n]{|c_n|} < \frac{1}{2|z_0|}$.

so $\sqrt[n]{|c_n|}\,|z_0| < \frac{1}{2}$ so $|c_n|\,|z_0|^n = |c_n z_0^n| < \frac{1}{2}^n$ converges by comparison test with geometric series, so $\sum c_n z^n$ absolutely convergent. $\square$.

case 3 $\ell \ne 0, \ell \ne \infty$. want: $R = \frac{1}{\ell}$, for all $\epsilon > 0$, there is an N s.t. $\sqrt[n]{|c_n|} < \ell + \epsilon$ for all $n > N$.

- assume $|z_1| < \frac{1}{\ell}$: set $\epsilon = \dfrac{1 - \ell |z_1|}{2|z_1|}$.

$\sqrt[n]{|c_n|} < l+\epsilon \Rightarrow \sqrt[n]{|c_n|} < l + \frac{l - l|z_1|}{2|z_1|} = \frac{1+l|z_1|}{2|z_1|}$

$\sqrt[n]{|c_n|}\,|z_1| < \frac{1+l|z_1|}{2} = q < 1 \leadsto |c_n z_1^n| < q^n$

converges □.

· pick z_2 s.t. $|z_2| > \frac{1}{l}$: for any $\epsilon>0$, $\exists$ ~~n≥~~ N, $\sqrt[n]{|c_n|} > l-\epsilon$ for infinitely many values of $n>N$, pick $\epsilon = \frac{l|z_2|-1}{|z_2|}$, so $\sqrt[n]{|c_n|} > l - \frac{l|z_2|-1}{|z_2|} = \frac{1}{|z_2|}$.

so $|c_n z^n| > 1$, so $|c_n z^n| \not\to 0$. □.

Examples ① $1+z+z^2+z^3+\dots$ $\quad \overline{\lim_{n\to\infty}}\sqrt[n]{1} = \lim_{n\to\infty} 1 = 1$ $\quad l=1$, $R=1$.

~~$1+\frac{z}{1}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots \lim_{n\to\infty}$~~ ② $1+z+z^4+z^9+z^{16}+\dots$ $\quad \overline{\lim_{n\to\infty}}\sqrt[n]{\begin{smallmatrix}0\\1\end{smallmatrix}}$ ← two limit pts 0, 1.

so $\overline{\lim_{n\to\infty}}\sqrt[n]{|c_n|} = 1$ $\quad l=1$, $R=1$.

③ $1+\frac{z}{1^s}+\frac{z^2}{2^s}+\frac{z^3}{3^s}+\dots+\frac{z^n}{n^s}+\dots$ $(s\geq 0)$ $\quad \sqrt[n]{|c_n|} = \frac{1}{n^{s/n}} = \frac{1}{e^{s\log n/n}}$.

$\frac{\log n}{n}\to 0$ as $n\to\infty$, so $\sqrt[n]{|c_n|}\to 1$ so $l=1$, $R=1$.

④ $1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$ $\quad l=0$, $R=\infty$. **note**: $(n!)^2 = \underbrace{(1\cdot n)}_{\geq n}(2\cdot n-1)\cdots\underbrace{(n\cdot 1)}_{\geq n}$.

$k(n-k+1)-n = (k-1)(n-k)\geq 0$ so each term $\geq n$

so $(n!)^2\geq n^n$, so $n!\geq\sqrt{n}^{\,n}$, so $\sqrt[n]{n!}\geq\sqrt{n}$.

so $\sqrt[n]{|c_n|} = \sqrt[n]{1/n!} \leq \frac{1}{\sqrt n}\to 0$ as $n\to\infty$. so $\overline{\lim}\sqrt[n]{|c_n|} = 0$.

⑤ $1-\frac{z^2}{2!}+\frac{z^4}{4!}+\dots$ $\quad z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots$ ← $R=\infty$. $\quad 1+z+2!z^2+3!z^3+\dots$ $R=0$

Behaviour on $|z|=R$ can be arbitrary (s-series).

$s=0$: $1+z+z^2+z^3+\dots$ ← diverges on $|z|=R$

$s=1$: $1+z+\frac{z^2}{2}+\frac{z^3}{3}+\dots$ ← diverges at some points of $|z|=R$, converges at others.

$s=2$: $1+z+\frac{z^2}{2^2}+\frac{z^3}{3^2}+\dots$ ← converges on $|z|=R$