but if z_{n_1} is first missing term, then $|s-(z_1+\dots+z_n)| \leq |s|-|s_{n_1}|$ $\to 0$ as $n \to \infty$ $\square$

Thm (Rearrangement of series) The terms of an absolutely convergent series can be rearranged without changing its sum.

Proof Let $\sum z_i = s$ be absolutely convergent, and let $z_{n_1}+z_{n_2}+\cdots$ be any rearrangement. Apply previous Lemma with $Z_i = z_{n_i}$ $\square$.

Products of series.

$$(z_1+z_2+z_3+\cdots)$$
$$(z_1'+z_2'+z_3'+\cdots)$$

$$= z_1 z_2 + (z_1 z_2' + z_2 z_1') + \cdots + (z_1 z_n' + z_2 z_{n-1}' + \cdots + z_n z_1') + \cdots$$

Remark for conditionally convergent sequences, product may not be convergent.

Thm (Multiplication of series) If $\sum z_i = s$, $\sum z_i' = s'$ are absolutely convergent, then their product is absolutely convergent, and sums to ss'.

Proof product series is $z_1 z_2 + z_1 z_2' + z_2 z_1' + \cdots$

claim: this is absolutely convergent,

proof: consider $|z_1 z_1'| + |z_1 z_2'| + \cdots + |z_j z_k'|$ $(*)$

let n be largest subscript appearing, then $(*) \leq (|z_1|+\cdots+|z_n|)(|z_1'|+\cdots+|z_n'|)$
$\leq \sigma\sigma'$

where $\sigma = \sum |z_i|$, $\sigma' = \sum |z_i'|$.

$\Rightarrow$ product series is absolutely convergent.

now consider following rearrangement:

$$z_1 z_1' + z_1 z_2' + z_1 z_3' + \cdots = z_1 s'$$
$$z_2 z_1' + z_2 z_2' + z_2 z_3' + \cdots = z_2 s'$$
$$z_3 z_1' + z_3 z_2' + \cdots = z_3 s'$$

so sum is $s'(z_1+z_2+\cdots) = ss'$. $\square$.

Series of functions

sequence of functions $f_n(z)$ Q: if f_n cts and $\lim_{n\to\infty} f_n(z)$ exists for all z, is $f(z)=\lim_{n\to\infty} f_n(z)$ cts?

A: no. Example $f_n(z) = z^n$. $\lim_{n\to\infty} f_n(z) = \begin{cases} 0 & |z|<1 \\ 1 & z=1 \\ \text{undef} & |z|\geq 1,\ \neq 1 \end{cases}$

Q: when does a sequence of functions converge to a cts function?

A: need uniform ~~continuity~~ convergence

Defn $\sum_{n=1}^{\infty} f_n(z)$ has partial sums $s_n(z) = f_1(z) + \dots + f_n(z)$. Assume $\sum_{n=1}^{\infty} f_n(z)$ converges to $s(z)$ for all z in $E \subseteq \mathbb{C}$. Then the sum is uniformly convergent in E if for any $\epsilon > 0$ there is an $N(\epsilon) > 0$ s.t. $|s_n(z) - s(z)| < \epsilon$ for all $n > N$ and all $z \in E$.

key fact N independent of z, i.e. same N works for all $z \in E$.

example · $s_n(z) = z^n$ ← not uniformly convergent in $|z| < 1$

check: $|s_n(z) - s(z)| = |z^n| < \epsilon \Rightarrow |z|^n < \epsilon \quad n \log|z| < \log(\epsilon)$

$n > \dfrac{\log(1/\epsilon)}{\log(1/z)} \to \infty$ as $|z| \to 1$, so there's no n that works for all z.

· $s_n(z) = z^n$ ← now consider this on $|z| \leq r < 1$, as above, can now choose n to be $\dfrac{\log(1/\epsilon)}{\log(1/r)}$ ← same fixed number works for all $z \in E$.

Thm If $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent in $E \subseteq \mathbb{C}$ and every f_n is cts at $z_0 \in E$, then $s(z) = \sum_{n=1}^{\infty} f_n(z)$ is also cts at z_0.

Proof suppose $z_0 + h \in E$, then $s(z_0+h) - s(z_0)$

$= s(z_0+h) - s_N(z_0+h) + s_N(z_0+h) - s_N(z_0) + s_N(z_0) - s(z_0)$

so $|s(z_0+h) - s(z_0)| \leq |s(z_0+h) - s_N(z_0+h)| + |s_N(z_0+h) - s_N(z_0)| + |s_N(z_0) - s(z_0)|$

by uniform continuity, given $\epsilon/3 > 0$, there is N s.t. $|s_N(z) - s(z)| < \epsilon/3$ for all $z \in E$.

so $|s(z_0+h) - s(z_0)| < \dfrac{\epsilon}{3} + |s_N(z_0+h) - s_N(z_0)| + \dfrac{\epsilon}{3}$.

cts, so $\exists\ \delta > 0$ s.t. $|s_N(z_0+h) - s_N(z_0)| < \frac{\epsilon}{3}$ for all $|h| < \delta$.

so $|s(z_0+h) - s(z_0)| < \epsilon$ for all $|h| < \delta \Rightarrow$ cts. $\square$.

Q: how do you know if $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent?

Thm consider $\sum_{n=1}^{\infty} f_n(z)$, and $|f_n(z)| \le a_n$ $n \in \mathbb{N}$ for all $z \in E$ where $\sum_{n=1}^{\infty} a_n$ converges. Then $\sum f_n(z)$ is uniformly convergent in E.

Proof Assume $\sum a_n$ converges. Then $\sum |f_n(z)|$ converges for all z by comparison test, i.e. $\sum f_n(z)$ is absolutely convergent for all $z \in E$. Furthermore

$$|s(z) - s_n(z)| \le |f_{n+1}(z) + f_{n+2}(z) + \dots| \le |f_{n+1}(z)| + |f_{n+2}(z)| + \dots$$

$$\le \underbrace{a_{n+1} + a_{n+2} + \dots}_{\text{doesn't depend on } z} < \epsilon \text{ for } N \text{ sufficiently large.}$$

so $s(z)$ is uniformly convergent in E. □.

Example $\sum f_n(z)$ $\quad f_n(z) = z^n - z^{n-1}$ so $s_n(z) = z^n$ on $|z| \le r < 1$.

$$|f_n(z)| = |z^n - z^{n-1}| = |z^{n-1}||z-1| \le r^{n-1}(r+1)$$

$\sum_{n=1}^{\infty} r^{n-1}(r+1) = (r+1)\frac{1}{1-r}$ converges, so $\sum f_n(z)$ uniformly convergent on $|z| \le r < 1$.

Integration

finite sums $\int_C f_1(z) + \dots + f_n(z)\,dz = \int_C f_1(z)\,dz + \dots + \int_C f_n(z)\,dz$. infinite sums:

Thm let $s(z) = \sum_{n=1}^{\infty} f_n(z)$ sum of cts functions on C then

$$\int_C s(z)\,dz = \int_C f_1(z)\,dz + \int_C f_2(z)\,dz + \dots$$ as long as $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on C

Proof $\sum f_n(z)$ uniformly convergent $\Rightarrow$ $s(z)$ cts $\Rightarrow$ integrable.

consider $s_n(z) = f_1(z) + \dots + f_n(z)$ uniform convergence: for any $\epsilon > 0$ there is N s.t. $|s(z) - s_n(z)| < \epsilon$ for all $n \ge N$. but then $\left|\int_C s(z) - s_n(z)\,dz\right| \le \epsilon l$

l = length of C. Therefore $\lim_{n\to\infty} \left|\int_C s(z) - s_n(z)\,dz\right| = 0 \Rightarrow \lim_{n\to\infty} \int_C s(z) - s_n(z)\,dz = 0$

Therefore: $\int_C s(z)\,dz = \lim_{n\to\infty} \int_C s_n(z)\,dz$, as required. □.

Q: what about analytic functions?

Thm (Weierstrass) $s(z) = \sum_{n=1}^{\infty} f_n(z)$, all $f_n(z)$ analytic, and uniformly convergent in every closed bounded domain $\bar{D} \subseteq C$. Then $s(z)$ is analytic in C. Furthermore, can differentiate term by term: $s_n^{(k)}(z) = f_1^{(k)}(z) + f_2^{(k)}(z) + \dots$ and uniformly convergent in every $\bar{D} \subseteq C$.

Proof  $z_0 \in G$ γ_R circle of radius R about z_0 s.t. $\gamma_R \subset G$

assume: $s(z) = \sum f_n(z)$ uniformly convergent.

then each of
$$\frac{k!}{2\pi i}\frac{s(z)}{(z-z_0)^{k+1}} = \frac{k!}{2\pi i}\frac{f_1(z)}{(z-z_0)^{k+1}} + \frac{k!}{2\pi i}\frac{f_2(z)}{(z-z_0)^{k+1}} + \cdots$$

is uniformly convergent as $\left|\frac{k!}{2\pi i}\frac{f_i(z)}{(z-z_0)^{k+1}}\right| \leq \frac{k!}{2\pi R^{k+1}}|f_i(z)|$.

so we can integrate term by term:
$$\frac{k!}{2\pi i}\int_{\gamma_R}\frac{s(z)}{(z-z_0)^{k+1}}dz = \frac{k!}{2\pi i}\int_{\gamma_R}\frac{f_1(z)}{(z-z_0)^{k+1}}dz + \frac{k!}{2\pi i}\int_{\gamma_R}\frac{f_2(z)}{(z-z_0)^{k+1}}dz + \cdots$$

$k=0$: $\int_{\gamma_R}\frac{s(z)}{(z-z_0)^{k+1}}dz = f_1(z) + f_2(z) + \cdots = s(z)$

i.e. get Cauchy's formula holds $\Rightarrow$ $s(z)$ analytic and infinitely differentiable

$k>0$: $s^{(k)}(z_0) = \frac{k!}{2\pi i}\int_{\gamma_R}\frac{s(z)}{(z-z_0)^{k+1}}dz$ and $f_n^{(k)}(z_0) = \frac{k!}{2\pi i}\int_{\gamma_R}\frac{f_n(z)}{(z-z_0)^{k+1}}dz$

remains to check uniform convergence of differentiated series. $\square$.

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§7 Power series

A <u>power series</u> is an infinite sum of the form
$$c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots$$
$c_i \in \mathbb{C}$. a is called the center.

$a=0$: $c_0 + c_1 z + c_2 z^2 + \cdots$

The <u>region of convergence</u> is the subset of $\mathbb{C}$ for which the sum converges.

Fact: always contains $\{a\}$.

Fact: may be equal to $\{a\}$, e.g. $1 + 2z + 2!z^2 + 3!z^3 + \cdots$

<u>Lemma</u> suppose $\sum c_n z^n$ converges at $z_0 \neq 0$. Then it is absolutely convergent for all $|z| < |z_0|$.