Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be two harmonic functions in G, which satisfy the Cauchy-Riemann equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for all points in G. Then u and v are said to be <u>conjugate</u> harmonic functions on G. u is the harmonic conjugate of v.
v u

<u>Thm</u> Let $f(z) = u(x,y) + iv(x,y)$ be a complex function defined in G. Then $f(z)$ is analytic in G iff u and v are conjugate harmonic functions.

<u>Proof</u> $\Rightarrow$ suppose f analytic in G, then u, v differentiable and satisfy Cauchy-Riemann equations, and $f'(z)$ also analytic, so

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$ $\leftarrow$ so all these functions differentiable in G, and satisfy CR equations.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \qquad \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right).$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

so u, v satisfy Laplace in G, $f''(z)$ analytic so 2nd derivatives exist and are cts, differentiable etc.

<u>$\Leftarrow$</u> u, v conjugate harmonic functions, then satisfy CR, and have cts 2nd order derivatives $\Rightarrow$ $f(z) = u + iv$ is analytic $\square$.

<u>Example</u> $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$u = x^2 - y^2$ $v = 2xy$.

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2y$ $\frac{\partial v}{\partial x} = 2y$ $\frac{\partial v}{\partial y} = 2y$ $\frac{\partial^2 u}{\partial x^2} = 2$ $\frac{\partial^2 u}{\partial y^2} = -2$.

$\frac{\partial^2 v}{\partial x^2} = 0$ $\frac{\partial^2 v}{\partial y^2} = 0$

<u>Fact</u>: given u, can find v.

Thm: u harmonic in G simply connected, define

$$v = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + c \qquad (c \text{ const})$$

then v is the harmonic conjugate of u.

Proof if v exists, then $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$

as v satisfies CR. existence of v holds if ↑ is exact differential, which it is as it follow from $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$ ← Laplace equations.

but then $v = \int_{(x_0,y_0)}^{(x,y)} dv + c$ □.

§6 Series

sequence (z_n) series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots$ infinite sum

partial sum: $s_n = z_1 + z_2 + \cdots + z_n$ ⇝ sequence of partial sums (s_n).

Defn The series converges if the sequence of partial sums converges. otherwise it diverges. Properly divergent if $|s_n| \to \infty$, otherwise oscillating.

Recall geometric series $1 + q + q^2 + \cdots = \frac{1}{1-q}$ if $|q| < 1$

same proof works for $q \in \mathbb{C}$!

Thm If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$

Warning: converse does not hold. Example?

Proof note $z_n = s_n - s_{n-1}$ so $\lim_{n\to\infty} z_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0$. □.

Absolute vs conditional convergence

If $z_1 + z_2 + z_3 + \cdots$ is a complex series, then $|z_1| + |z_2| + |z_3| + \cdots$ is a real series.

Thm If $\sum |z_n|$ converges, then $\sum z_n$ converges. (absolute convergence ⇒ convergence).

Proof assume $\sum |z_n|$ converges, let $s_n = z_1 + z_2 + \cdots + z_n$
$\sigma_n = |z_1| + |z_2| + \cdots + |z_n|$

$$|s_{n+p} - s_n| = |z_{n+1} + \cdots + z_{n+p}| \le |z_{n+1}| + \cdots + |z_{n+p}| = \sigma_{n+p} - \sigma_n$$

recall Cauchy convergence criterion: (σ_n) converges iff for all $\epsilon > 0$, there is $N(\epsilon)$ s.t. $|\sigma_{\cancel{n+p}\,a} - \sigma_b| < \epsilon$ for all $a, b \geq N$.

so for all $\epsilon > 0$ there is N s.t. $|\sigma_{n+p} - \sigma_n| < \epsilon$ for all $n \geq N, p \geq 0$

so $|s_{n+p} - s_n| \le \epsilon$ for all $n \geq N, p \geq 0$ $\Rightarrow$ (s_n) converge $\Rightarrow$ $\sum z_n$ converges. $\square$

Examples If $\sum z_n$ is not absolutely convergent, it is conditionally convergent

① $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ not absolutely convergent why?
but convergent. why? so conditionally convergent.

② $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ absolutely convergent. why?

Thm (Addition/subtraction of series) If $\sum z_n$ and $\sum z_n'$ are convergent, with sum s and s', then $\sum (z_n + z_n') = s + s'$
$\sum (z_n - z_n') = s - s'$.

Proof same as real analysis/calc 2 $\square$.

Absolutely convergent series can be rearranged, and they still converge to the same thing

Lemma Let $\sum z_n$ be absolutely convergent (to z). Consider a (possibly infinite) family of series

Z_1 $\quad z_{1,1} + z_{1,2} + z_{1,3} + \cdots$
Z_2 $\quad z_{2,1} + z_{2,2} + \cdots$
$\vdots$ $\quad z_{3,1} + \cdots$

} where each z_i appears exactly once in this collection.

then each subseries is absolutely convergent, to Z_i say, and $Z_1 + Z_2 + \cdots$ is absolutely convergent to Z.

Proof $\sum z_n$ absolutely convergent to s say $\Rightarrow$ each Z_i absolutely convergent, and in fact $\sum Z_i$ abs convergent

further $Z_1 + \cdots + Z_n \le s$ for all n

as every z_i appears in some Z_n, $|s - (Z_1 + Z_2 + \cdots Z_n)|$
$\le |z_{n_1}| + \cdots$ ← sum of missing terms.