want to show for any $\epsilon > 0$, there is $\delta > 0$ s.t. $\left| \int_{\gamma_R} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| < \epsilon$ $\circledast$

for all $R < \delta$

$\circledast$: $\left| \int_{\gamma_R} \frac{f(z)}{z-z_0} dz - f(z_0) \int_{\gamma_R} \frac{dz}{z-z_0} \right| = \left| \int_{\gamma_R} \frac{f(z)-f(z_0)}{z-z_0} dz \right|$

$f(z)$ is cts at z_0, so $\exists\ \delta > 0$ s.t. $|f(z) - f(z_0)| < \frac{\epsilon}{2\pi}$ for all $|z-z_0| < \delta$.

so $\circledast < \frac{1}{R} \frac{\epsilon}{2\pi} 2\pi R = \epsilon$, as required $\square$.

Note if z_0 not in C, $\frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz = 0$.

§5.7. Infinite differentiability

Thm If $f(z)$ is analytic in a domain G, then $f(z)$ is infinitely differentiable in G, i.e. $f(z)$ has derivatives of all orders in G, and the nth derivative is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad z_0 \in G,\ n \in \mathbb{N}.$$

C piecewise smooth Jordan curve inside G, contains z_0.

Proof (induction on n.) base case $n=1$, we've just done this.

assume true for $n-1$. consider

$$f^{(n)}(z_0) = \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h}.$$

← know $n-1$ so can replace this with integral around C.

← can replace integral around C w/ integral around γ_R.

$$= \frac{(n-1)!}{2\pi i h} \int_{\gamma_R} f(z) \left(\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0)^n} \right) dz$$

$$= \frac{(n-1)!}{2\pi i h} \int_{\gamma_R} f(z) \frac{(z-z_0)^n - (z-z_0-h)^n}{(z-z_0-h)^n (z-z_0)^n} dz$$

Fact: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$

$$= \frac{(n-1)!}{2\pi i \cancel{h}} \int_{\gamma_R} f(z) \frac{\cancel{h}\left((z-z_0)^{n-1} + (z-z_0)^{n-2}(z-z_0-h) + \dots + (z-z_0-h)^{n-1}\right)}{(z-z_0-h)^n (z-z_0)^n}.$$

now consider

$$\left| \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} - \frac{n!}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right|$$

let $M = \max_{z\in\gamma_R} |f(z)|$.

$$= \left| \frac{(n-1)!}{2\pi i} \int_{\gamma_R} f(z)\, \frac{(z-z_0)^n + (z-z_0)^{n-1}(z-z_0-h) + \dots + (z-z_0)(z-z_0-h)^{n-1} - n(z-z_0-h)^n}{(z-z_0-h)^n (z-z_0)^{n+1}} \right.$$

$$\le \frac{(n-1)!}{2\pi}\, 2\pi R M |h| \, \frac{(2R)^{n-1} + 2(2R)^{n-1} + \dots + n(2R)^{n-1}}{(R-|h|)^n R^{n+1}} \to 0 \text{ as } R \to 0$$

using ~~(reverse)~~ triangle inequality:

$$R - |h| = \big| |z-z_0| - |h| \big| \le |z-z_0-h| \le |z-z_0| + |h| < 2R.$$

So $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$, as required. $\square$.

Corollary If $f(z)$ is analytic in a domain G, then so are all its derivatives, $f'(z), f''(z), \dots$

Remark: totally different from real case!

Converse: Thm (Morera) Let $f(z)$ be cts in a domain G, and let $\int_c f(z)\,dz = 0$ for every piecewise smooth curve c in G. Then $f(z)$ is analytic in G.

Proof let $F(z) = \int_{z_0}^{z} f(w)\,dw$, defines an analytic function in G, with derivative $F'(z) = f(z)$, but F analytic $\Rightarrow$ $F' = f$ analytic. $\square$.

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§5.8 Harmonic functions

$u: \mathbb{R}^2 \to \mathbb{R}$ $u(x,y)$ is harmonic in a domain G if it has cts second partial derivatives $\frac{\partial^2 u}{\partial x^2}$ $\frac{\partial^2 u}{\partial x \partial y}$ $\frac{\partial^2 u}{\partial y \partial x}$ $\frac{\partial^2 u}{\partial y^2}$ at every point in G and satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ everywhere in G.