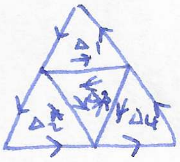
Proof (of claim). Δ triangle in G. Suppose $\left|\int_\Delta f(z)dz\right| = M > 0$.

subdivide into congruent triangles



$\int_\Delta = \int_{\Delta_1} + \int_{\Delta_2} + \int_{\Delta_3} + \int_{\Delta_4}$. ← at least one of these integrals has $\left|\int_{\Delta_i}\right| \geq \frac{M}{4}$

now subdivide that triangle:  get $\Delta_i^{(2)}$ with $\left|\int_{\Delta_i^{(2)}} f(z)dz\right| \geq \frac{M}{4^2}$.

⇝ get sequence of nested triangles $\Delta_i^{(1)} \supset \Delta_i^{(2)} \supset \Delta_i^{(3)} \supset \cdots$

with $\left|\int_{\Delta_i^{(k)}} f(z)dz\right| \geq \frac{M}{4^k}$ ↑ note this uses G simply connected.

note if perimeter of $\Delta = \ell$, then perimeter of $\Delta_i^{(k)}$ is $\frac{\ell}{2^n}$.

claim $\exists$ unique point $z_0 \in G$ in $\bigcap \Delta_i^{(k)}$ (analogous with nested rectangles) □

f analytic $\Rightarrow$ derivative $f'(z_0)$ is finite, so for all $\epsilon > 0$, there is $\delta > 0$ s.t.

$|z-z_0| < \delta \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon. \Leftrightarrow |f(z)-f(z_0)-f'(z_0)(z-z_0)| < \epsilon|z-z_0|$

note: $\int_{\Delta_i^{(k)}} f(z)-f(z_0)-(z-z_0)f'(z)dz = \int_{\Delta^{(k)}} f(z)dz - f(z_0)\underbrace{\int_{\Delta^{(k)}} dz}_{0} - f'(z_0)\underbrace{\int_{\Delta^{(k)}} z\, dz}_{0}$

$+ z_0 f'(z)\underbrace{\int_{\Delta^{(k)}} dz}_{0}$.

$|z-z_0| < \delta$

$\Delta^{(k)}$ $\quad \ell_k$ = perimeter of $\Delta^{(k)}_i$.

so $|z-z_0| < \ell_k$.

so $|f(z)-f(z_0)-f'(z_0)(z-z_0)| < \epsilon \ell_k$.

so $\left|\int_{\Delta^{(k)}} f(z)dz\right| = \left|\int_{\Delta^{(k)}} f(z)-f(z_0)-(z-z_0)f'(z_0)\, dz\right| < \epsilon \ell_k^2 = \frac{\epsilon \ell^2}{4^k}$

so $\frac{M}{4^k} \leq \left|\int_{\Delta^{(k)}} f(z)dz\right| \leq \frac{\epsilon \ell^2}{4^k}$ ← holds for all $\epsilon > 0$

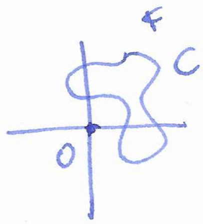
$\Rightarrow M = 0$ □.

Multiple curves C_0 contains $C_1, \ldots, C_n$ with disjoint interiors, $f(z)$ analytic in $\text{int}(C_0) \setminus \text{int}(C_i)$

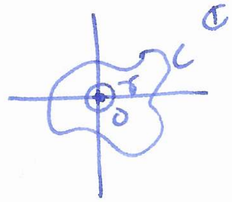
claim: $\int_{C_0} f(z)dz = \int_{C_1} f(z)dz + \cdots + \int_{C_n} f(z)dz$

proof: add dividing arcs □.

Example $f(z) = \frac{1}{z}$, $f'(z) = \frac{-1}{z^2}$ analytic except at $z=0$. C Jordan curve.



C does not enclose 0, $\int_C \frac{1}{z}\,dz = 0$



suppose C does include 0.
Cauchy's Thm does not apply!

but: $\int_C \frac{1}{z}\,dz = \int_\gamma \frac{1}{z}\,dz$ where γ is a small circle enclosing 0

parameterize γ: $\gamma(\theta) = R\cos\theta + iR\sin\theta$ $\quad \gamma'(t) = -R\sin\theta + iR\cos\theta$

$$\int_0^{2\pi} \frac{1}{R\cos\theta + iR\sin\theta} \cdot (-R\sin\theta + iR\cos\theta)\,d\theta = \int_0^{2\pi} \frac{-\sin\theta + i\cos\theta\,(\cos\theta - i\sin\theta)}{\cos\theta + i\sin\theta\,(\cos\theta - i\sin\theta)}\,d\theta$$

$$= \int_0^{2\pi} \cancel{-\sin\theta\cos\theta} + \cancel{\sin\theta\cos\theta} + i(\cos^2\theta + \sin^2\theta)\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i$$

slicker $\gamma(\theta) = z(\theta) = R\cos\theta + iR\sin\theta$ $\quad dz = -R\sin\theta + iR\cos\theta\,d\theta$
$\quad = i(R\cos\theta + i\sin\theta)\,d\theta$.

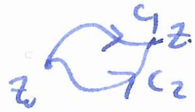
so $\int_\gamma \frac{dz}{z} = \int_\gamma i\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i$.

More generally consider $f(z) = \frac{1}{z - z_0}$.
- if C encloses z_0 $\int_C \frac{1}{z-z_0}\,dz = 2\pi i$
- if C doesn't contain z_0 $\int_C \frac{1}{z-z_0}\,dz = 0$.

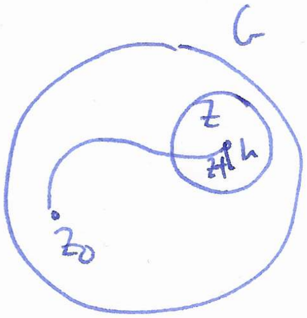
Indefinite integrals

Thm $f(z)$ analytic in simply connected domain G. Then $F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$ taken along any piecewise smooth curve from z_0 to z defines a single valued analytic function in G with derivative $F'(z) = f(z)$.

Proof well defined: Cauchy's Thm.



check derivative:



K nbd of z in G. $z+h \in K$

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw = \int_{z}^{z+h} f(w)\,dw$$

just choose path of integration here to be a straight line in K.

then $\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(w)\,dw - f(z)$ ← $= f(z).\frac{1}{h}\int_z^{z+h} dz$

$$= \frac{1}{h}\int_z^{z+h} f(w) - f(z)\,dz$$

f cts, so for all $\epsilon > 0$, there is $\delta > 0$ s.t. $|z-w| < \delta \Rightarrow |f(w) - f(z)| < \epsilon$,

so $\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| < \frac{\epsilon}{h}.h = \epsilon$, so $F'(z) = f(z)$, as required □.

Remark: ① only uses f cts. ② what if G not simply connected?

Def^n A single valued function $F(z)$ s.t. $F'(z) = f(z)$ in a domain G is called an indefinite integral (or antiderivative) for f.

Th^m Every indefinite integral of $f(z)$ has the form $H(z) = F(z) + c$ $= \int_z^{z_0} f(z)\,dz + c$ for some $c \in \mathbb{C}$.

Proof Suppose F, H are two indefinite integrals for f. Then $F'(z) - H'(z) = 0$

$(F-H)'(z) = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = 0$ … etc. so $(F-H)(z) = $ const. □.

§5.6 Cauchy's integral formula

Th^m $f(z)$ analytic in a domain G containing a piecewise smooth Jordan curve, and its interior. Then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$ if z_0 lies in C

Proof $z_0 \in C$, so $\frac{f(z)}{z - z_0}$ analytic everywhere in C, except at z_0.

Let γ_R be circle of radius R about z_0, R small enough so this is contained in C

Then $\int_C \frac{f(z)}{z - z_0}\,dz = \int_{\gamma_R} \frac{f(z)}{z - z_0}\,dz$ ← doesn't depend on R!

$= \lim_{R \to 0} \int_{\gamma_R} \frac{f(z)}{z - z_0}\,dz$ claim: $\lim_{R \to 0} \int_{\gamma_R} \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$.