Thm $f: \mathbb{C} \to \mathbb{C}$ ($G \to \mathbb{C}$) differentiable/analytic and $f'(z) \neq 0$ then f is conformal.

Examples $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$. $f'(z) = 2z \neq 0$ except if $z=0$ so f is conformal on $\mathbb{C}\setminus\{0\}$

note: $z \mapsto z^2$ not conformal at $z=0$ as $\arg(z^2) = 2\arg(z)$ so doubles all angles at 0.

Defn If $f: \mathbb{C} \to \mathbb{C}$ preserves angles, but not nec. orientations, we say it is isogonal.

Example $f: \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ is isogonal, but not conformal. (in fact reverses orientation).

Remark: $\arg(f'(z))$ = amount of rotation at z. $|f'(z)|$ amount of expansion at z.

§5 Integration (along curves)

Let $I \subseteq \mathbb{R}$ and let C be a smooth curve, i.e. a map $z: I \to \mathbb{C}$, $t \mapsto z(t)$.
then we can integrate a complex function along C:

Riemann sums: Choose a partition $z_0, z_1, \dots, z_n$ along C.
and take $\sum_{i=1}^{n} f(z_i')\Delta z_i$ where $z_i' \in$ arc from z_{i-1} to z_i
$\Delta z_i = z_i - z_{i-1}$.

Let l_i = length of arc from z_{i-1} to z_i and set $\lambda = \max_i l_i$

suppose $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(z_i')\Delta z_i$ exists over all subdivisions, then we say that f is integrable along C, and write $\int_C f(z)dz$ for the limit.

Thm If $f(z)$ is continuous in a domain G containing a smooth curve C, then $f(z)$ is integrable along C.

Proof Let $z_i = x_i + iy_i$, $z_i' = x_i' + y_i' i$, $\Delta z_i = \Delta x_i + \Delta y_i i$, $f(z) = u(x,y) + iv(x,y)$

Then $\sum f(z_i')\Delta z_i = \sum(\underset{u(x_i',y_i')}{u_i} + i\underset{v(x_i',y_i')}{v_i})(\Delta x_i + i\Delta y_i) = \underbrace{\sum u_i\Delta x_i - v_i \Delta y_i}_{} + i\underbrace{(\sum v_i \Delta x_i + u_i \Delta y_i)}_{}$

$\xrightarrow[\lambda \to 0]{\text{as}} \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$

so $\int_C f(z)dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$ $\square$.

Remark ① $\int_C f(z)dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$

$$= \int_C (u+iv)(dx+idy)$$

② $C : [a,b] \to \mathbb{C}$ $\qquad z(t) = x(t) + iy(t)$

$$\int_C f(z)dz = \int_C f(z(t))\,z'(t)\,dt \qquad f(z) = u(z)+iv(z)$$

$$= \int_C \big(u(z(t)) + iv(z(t))\big)\big(x'(t) + iy'(t)\big)\,dt.$$

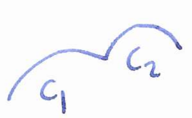
$$= \int_C u(z(t)).\,x'(t) - v(z(t))y'(t)\,dt$$

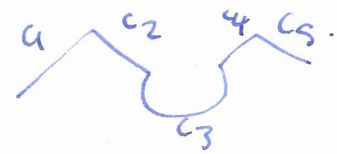
$$+ i\int_C v(z(t))x'(t) + u(z(t))\,y'(t)\,dt$$

$$= \int_C R(t)\,dt + \int_C I(t)\,dt \quad \text{where} \quad \begin{aligned} R(t) &= \operatorname{Re} f(z(t))z'(t) \\ I(t) &= \operatorname{Im} f(z(t))z'(t) \end{aligned}$$

Defn A curve $C : I \to \mathbb{C}$ is <u>piecewise smooth</u> if there is a subdivision of $I = [a,b]$ into $\underset{a}{\overset{\shortparallel}{[a_0}},a_1] \cup [a_1,a_2] \cup \cdots [a_{n-1},\underset{b}{\overset{\shortparallel}{a_n}}]$.

such that $c_j = c|_{[a_j,a_{j+1}]}$ is smooth and $c : I \to \mathbb{C}$ is $c\mathcal{B}$.

Example 

piecewise smooth 

- $\int_C f(z)dz = \int_{C_1} f(z)dz + \cdots + \int_{C_n} f(z)dz$

Example C piecewise smooth curve from z_0 to z_1

$$\int_C z^n\,dz = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$ as long as $n \neq -1$ and C does not hit 0 if $n < 0$.

check: $$\int_C z^n\,dz = \int_a^b z^n(t)\,z'(t)\,dt = \int_a^b \frac{1}{n+1}\frac{d}{dt}\left(z^{n+1}(t)\right)dt$$

$$= \left[\frac{1}{n+1} z^{n+1}(t)\right]_a^b = \frac{1}{n+1}\left(z^{n+1}(b) - z^{n+1}(a)\right) = \frac{1}{n+1}\left(z_1^{n+1} - z_0^{n+1}\right)$$

Remark: this does not depend ~~of~~ on C! all paths give same answer.

In particular, if C is a loop, $z_0 = z_1$ and $\int_C z^n\,dz = 0$

Useful properties

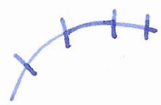
$C^- \leftarrow C$ with opposite direction.

$C: I \longrightarrow \mathbb{C}$.
$[a,b]$
$t \longmapsto z(t)$.

$C^-: t \longmapsto z(b+a-t)$
$[a,b]$

Thm $$\int_{C^-} f(z)\,dz = -\int_C f(z)\,dz$$

"Proof" $\int_C f(z)\,dz$



C $\quad \sum f(z_k)(z_{k+1} - z_k)$

C^- $\quad \sum f(z_k)(z_k - z_{k+1})$ $\square$.

Thm $f(z)$ cts on piecewise smooth C and $|f(z)| \leq M$ for all $z \in C$

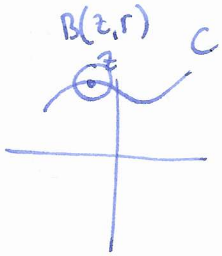
then $$\left|\int_C f(z)\,dz\right| \leq M\ell, \quad \ell = \text{length of } C.$$

Proof $$\left|\int_C f(z)\,dz\right| \approx \left|\sum f(z_k)(z_k - z_{k-1})\right| \leq \sum |f(z_k)|\,|z_k - z_{k-1}|$$
$$\leq M \sum |z_k - z_{k-1}|$$
$$\leq M\ell.$$

Approximation by polygonal curves

Lemma Let $f(z)$ be cts in a domain G, and C piecewise smooth curve $C \subseteq G$. For any $\epsilon > 0$ there is a polygonal curve L inscribed in C, contained in G, s.t. $\left| \int_C f(z)dz - \int_L f(z)dz \right| < \epsilon$.

Proof (sketch) claim: there is a bounded domain $C \subseteq D \subseteq G$ s.t. $\overline{D} \subseteq G$.



$\mathbb{C}$. for each point $z \in C$, there is a neighborhood $B(z,r) = \{w \mid |z-w| < r\}$ s.t. $z \in B(z,r) \subseteq G$.

· can choose $r < 1$ say

$\bigcup_{z \in C} B(z, \frac{r}{2}) = D$ is a domain; · ~~then diam~~ bounded as $\text{diam}(D) \leq \text{diam}(C) + 2$.

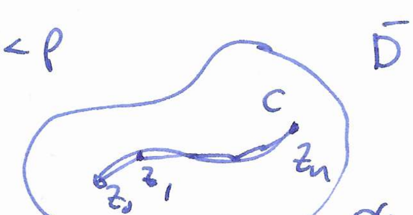
· $\overline{D} \subseteq \bigcup \overline{B(z, \frac{r}{2})} \subseteq G$ so $\overline{D} \subseteq G$.

· ~~choose by BZ, D~~ let ρ = distance from C to ∂D ($\rho > 0$)

$f(z)$ cts in G $\Rightarrow$ cts in $\overline{D}$ $\leftarrow$ closed bounded $\Rightarrow$ f is uniformly cts on $\overline{D}$.

so for any $\epsilon > 0$, there is a $\delta > 0$ s.t. for all $z_1, z_2 \in \overline{D}$ with $|z_1 - z_2| < \delta$

$|f(z_1) - f(z_2)| < \frac{\epsilon}{2\ell}$ ℓ = length of C, and wlog $\delta < \rho$



now partition C by $z_0, z_1, \ldots z_n$ w/ $|z_k - z_{k-1}| < \delta$

so straight line segments connecting these are in $\overline{D} \subset G$

~~now consider $\left| \int_C f(z)dz - \int_L f(z)dz \right| = \left| \sum_i \int_{\gamma_i \cup \overline{\sigma_1}} f(z)dz \right|$~~

$C = \bigcup \gamma_i$
$L = \bigcup \sigma_i$.

approx C: $\left| \int_C f(z)dz - \sum_i \int_{\sigma_i} f(z_i)dz \right| \leq \sum_i \int_{\gamma_i} |f(z) - f(z_i)| dz \leq \ell \cdot \frac{\epsilon}{2\ell} = \frac{\epsilon}{2}$

] (*)

approx L: $\left| \int_L f(z)dz - \sum_i \int_{\sigma_i} f(z_i)dz \right| \leq \sum_i \int_{\sigma_i} |f(z) - f(z_i)| dz \leq \ell \cdot \frac{\epsilon}{2\ell} = \frac{\epsilon}{2}$.

(*) these are equal as integrating a constant function only depends on endpoints. $\square$

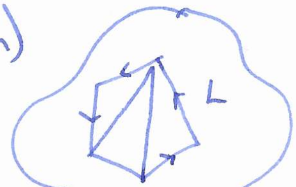
so triangle inequality gives $\left| \int_L f(z)dz - \int_C f(z)dz \right| \leq \epsilon$. $\square$

Approximation of polygonal closed curves by triangles.

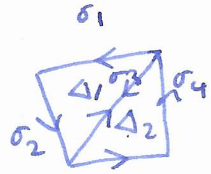
Fact Let $f(z)$ be cts in a simply connected domain G containing a closed polygonal curve L. Then there are triangles $\Delta_1, \dots, \Delta_n$ s.t. $\int_L f(z)dz$

$= \int_{\Delta_1} f(z)dz + \dots + \int_{\Delta_n} f(z)dz$

Proof (sketch)



note:

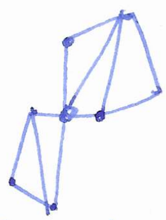


$\int_{\Delta_1} f(z)dz = \int_{\sigma_1} + \int_{\sigma_2} + \int_{\sigma_3}$

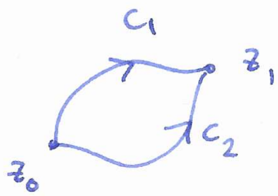
$\int_{\Delta_2} f(z)dz = \int_{\sigma_4} + \int_{\sigma_5} - \int_{\sigma_3}$.

so $\int_L = \int_{\Delta_1} + \int_{\Delta_2}$.

self intersections:
subdivide edges
at intersection, then divide regions. $\square$



§ Cauchy's integral theorem



If c_1, c_2 have same endpoints

and $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$, (*) then $\int_{c_1 \cup \overline{c_2}} f(z)dz = 0$

if (*) holds for all curves c_i connecting points z_0, z_1 then $\int_C f(z)dz = 0$

for all closed curves C.

Q: when does this happen? (know it happens for polynomials)

Thm (Cauchy's integral theorem) Let $f(z)$ be analytic in a simply connected domain G. Then $\int_C f(z)dz = 0$, for all piecewise smooth curves C contained in G.

Proof claim: suffices to know this for all triangles Δ: $\int_\Delta f(z)dz = 0$.

Proof (assuming claim): The piecewise smooth curve C can be approximated by a polygonal curve L with $\left| \int_L f(z)dz - \int_C f(z)dz \right| < \epsilon$ (⊗) Then $\int_L$ can be replaced by $\sum \int_{\Delta_i}$ integral over triangles. (⊗) holds $\forall \epsilon > 0 \Rightarrow \int_C f(z)dz = 0$ $\square$.