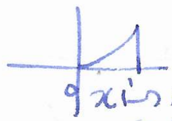
Uniform continuity

$f(z)$ defined on subset $E \subseteq \mathbb{C}$ is uniformly continuous if for all $\epsilon > 0$ there is a $\delta > 0$ s.t. for all $z, z' \in E$ $|z - z'| < \delta \Rightarrow |f(z) - f(z')| < \epsilon$.

Example $f: \mathbb{R} \to \mathbb{R}$.  $x \mapsto x^2$ on $[0,1]$

Non-example $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ on all of $\mathbb{R}$. $f: \mathbb{R} \to \mathbb{R}$ $(0,1)$ $x \mapsto \frac{1}{x}$ on $(0,1)$.

Fact f cts on G closed bounded $\Rightarrow$ f uniformly continuous.

Thm (Heine-Borel) $\bar{G}$ bounded closed domain. For each $z \in \bar{G}$ choose a disc D_z. Then there is a finite collection of discs $D_{z_1}, \dots, D_{z_n}$ which cover $\bar{G}$, i.e. $\bar{G} \subseteq \bigcup_{i=1}^{n} D_{z_i}$.

Proof $\bar{G}$ bounded $\Rightarrow$ contained in a rectangle R.  R suppose $\bar{G}$ ~~can~~ not covered by finitely many discs. divide R into four rectangles  at least one of $\bar{G} \cap R_i$ cannot be covered by finitely many rectangles. continue subdividing, get sequence of nested rectangle $R_1 \supseteq R_2 \supseteq R_3 \dots$ s.t. $\bar{G} \cap R_i$ ~~can~~ not covered by finitely many rectangles. but $\bigcap R_i = \{z\}$, and D_z has positive radius, so contains R_k for k sufficiently large, so $R_k \cap \bar{G}$ is covered by a single disc. #. $\square$.

Thm If f is cts on $\bar{G}$ closed bounded domain, then f is uniformly cts on $\bar{G}$.

Proof suppose f cts on $\bar{G}$, and let $z \in \bar{G}$ then for any $\frac{\epsilon}{2} > 0$ there is a disc D_z $|z - z'| < \delta$ s.t. $|f(z) - f(z')| < \frac{\epsilon}{2}$ for all $z \in D_z$. but then for any $z', z'' \in D_z$.

$|f(z') - f(z'')| \leq |f(z') - f(z) + (f(z) - f(z''))| \leq |f(z') - f(z)| + |f(z) - f(z'')| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

now replace D_z with $D_z' = |z - z'| < \delta/2$, i.e. disc of half the radius. By Heine-Borel, there is a finite collection $D'_{z_1} \dots D'_{z_n}$ which cover $\bar{G}$. Choose δ = min radius of $D'_{z_1} \dots D'_{z_n}$. Now let z', z'' be any two points w/ $|z' - z''| < \delta$. $z' \in D'_{z_k}$ for some k, and so z' and $z'' \in D_{z_k}$, so $|f(z') - f(z'')| < \epsilon$, as required. $\square$.

§4 Differentiation

<u>Defn</u> f defined on domain $G \subseteq \mathbb{C}$ is <u>differentiable</u> at $z \in G$ if the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ exists and is finite, <u>notation</u>: <u>$f'(z)$</u>.
$(z, z+\Delta z \in G)$. called the <u>derivative</u>.

<u>Warning</u> this is totally different from $f: \mathbb{R}^2 \to \mathbb{R}^2$ being differentiable!

<u>Defn</u> f is analytic on G if differentiable at every point of G, and analytic at z if differentiable in a nbd of z.

<u>example</u> $f: \mathbb{C} \to \mathbb{C}$ $f(z) = z^2$ is differentiable:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2\Delta z\, z + \Delta z^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z = 2z.$$

<u>Non-example</u> $f: \mathbb{C} \to \mathbb{C}$ $f(z) = \mathrm{Re}(z)$ $a+bi \mapsto a$. this is cts on $\mathbb{C}$, but not differentiable anywhere:

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\mathrm{Re}(z+\Delta z) - \mathrm{Re}(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\mathrm{Re}(z) + \mathrm{Re}(\Delta z) - \mathrm{Re}(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\mathrm{Re}(\Delta z)}{\Delta z}$$ ← this limit DNE! Δz $r \to 0$

$\Delta z = h$ $\quad \frac{\mathrm{Re}(h)}{h} = \frac{h}{h} = 1$.

$\Delta z = hi$ $\quad \frac{0}{hi} = 0 \neq 1$.

<u>Note</u> $f: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x,0)$ is differentiable as a real valued function as $\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$ exists

<u>Exercise</u> show $f(z) = \bar{z}$ is not (complex) differentiable.

standard derivative rules hold:

$(z^n)' = nz^{n-1}$

$\Rightarrow$ all polynomials are analytic

$\Rightarrow$ all rational functions $\frac{p(z)}{q(z)}$ are analytic when $q(z) \neq 0$.

$(cf(z))' = cf'(z)$

$(f(z)+g(z))' = f' + g'$

$(fg)' = f'g + fg'$

$\left(\frac{f}{g}\right)' = \frac{g'f - f'g}{g^2}$

$(f(g(z)))' = f'(g(z))\,g'(z)$.

Complex differentials

$f'(z)$ derivative is locally a $\mathbb{C}$-linear map $\mathbb{C}\to\mathbb{C}$.

define $df = f'(z)$ to be this family of local maps $\mathbb{C}\to\mathbb{C}$. at each $z\in\mathbb{C}$.

text: likes to write $w = f(z)$; and Δz for the local variable $\mathbb{C}\to\mathbb{C}$. $\Delta z \mapsto f'(z)\Delta z$

so $df = dw = f'(z)\Delta z$. note: if $f(z) = z$.
then $dz = (z)'\Delta z = \Delta z$.

so we can also write dz for the local variable, i.e. $df = f'(z)\,dz$.

Cauchy-Riemann Equations

consider a real function $u:\mathbb{R}^2\to\mathbb{R}$, derivative is $\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix}$. $\begin{bmatrix}\frac{\partial v}{\partial x}\\ \frac{\partial v}{\partial y}\end{bmatrix}$.
$v:\mathbb{R}^2\to\mathbb{R}$ derivative is

$f:\mathbb{C}\to\mathbb{C}$.
$z\mapsto f(z)$.
$x+iy \mapsto f(x+iy) = u(x,y)+v(x,y)i$ two real valued functions of 2 vars

Thm $f(z)$ is (complex) differentiable at $z=x+iy$ iff these real-valued functions u and v are differentiable at (x,y) and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. ← Cauchy Riemann equations.

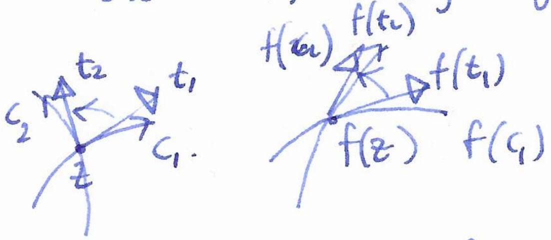
~~Proof~~ intuition: $M:\mathbb{R}^2\to\mathbb{R}^2$ linear map $M = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. $\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{bmatrix}$.
$M:\mathbb{C}\to\mathbb{C}$
$z\mapsto \lambda z$ $(x+iy)\mapsto (a+bi)(x+iy) = ax-by+(ay+bx)i$ $\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

□.

Fact $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Conformal maps

$f:\mathbb{C}\to\mathbb{C}$.
$z\mapsto f(z)$



f is conformal at z if for every pair of tangent vectors t_1, t_2 the angle between t_1 and t_2 is equal to the angle between $f(t_1)$ and $f(t_2)$ and f preserves orientation i.e. if t_2 is anticlockwise of t_1 then $f(t_2)$ is anticlockwise of $f(t_1)$.

locally: $f'(z):\mathbb{C}\to\mathbb{C}$. is a $\mathbb{C}$-linear map, which is conformal
$z\mapsto f'(z)z$ as long as $f'(z)\neq 0$.
$re^{i\theta}$