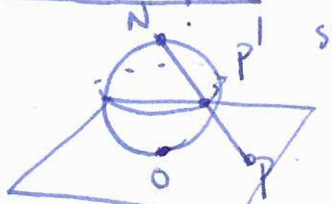
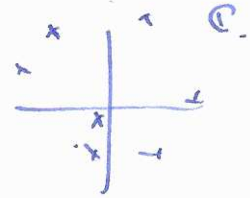
Riemann sphere



stereographic projection gives a bijection between $S \setminus N$ and $\mathbb{C}$.

we can think of N as a "point at infinity" for $\mathbb{C}$. so $S = \mathbb{C} \cup \{\infty\}$

Defn we say a sequence (z_n) approaches infinity $\lim_{n\to\infty} z_n = \infty$ if for any $M>0$ $\exists\ N(M)>0$ s.t. $|z_n| > M$ for all $n \geq N$.



Fact if $z_n \to \infty$ then $p(z_n)$ in S tends to N.

Remark the exterior of any circle centered at O is called a neighbourhood of ∞, so $z_n \to \infty$ iff any nbd of ∞ contains infinitely many elements of (z_n).
∞ is a limit point of (z_n)

§3 complex functions

Recall $f: A \to B$ is a function if the set of pairs $(a, f(a)) \subseteq A \times B$ satisfies: for each $a \in A$ is there is a unique pair $(a, f(a))$.

notation $a \mapsto f(a)$. $\quad f: A \to B$ $\quad f(A) = \{f(a) \in B \mid a \in A\}$.

domain ↗ (A), ↖ range / codomain (B), ↑ image ($f(A)$).

In this course we will consider $f: \mathbb{C} \to \mathbb{C}$.

$f: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.

Remark $\mathbb{C} \cong \mathbb{R}^2$ so $f: \mathbb{C} \to \mathbb{C}$ can be thought of as $f: \mathbb{R}^2 \to \mathbb{R}^2$.

Recall: $f: \mathbb{C} \to \mathbb{C}$ is injective or one-to-one, if $f(a) = f(b) \Rightarrow a = b$.

if $f: \mathbb{C} \to \mathbb{C}$ is injective then there is an inverse function $f^{-1}: \mathbb{C} \to \mathbb{C}$
$f(a) \mapsto a$.

in general f is not injective, there is no inverse function,
but the pre-image $f^{-1}(z) = \{w \in \mathbb{C} \mid f(w) = z\}$ is a subset of $\mathbb{C}$.

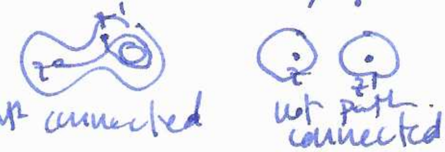
Example $f: \mathbb{C} \to \mathbb{C}$
$z \mapsto z^2$
$(a+bi) \mapsto (a+bi)^2 = a^2 - b^2 + 2abi$

two-to-one except at $0, \infty$.
$f^{-1}(z) = \{$two square roots of $z\}$.
Q: what does this look like geometrically?

Curves: a parameterised curve is a continuous map $I=[a,b]\to\mathbb{C}$.
$t\mapsto (x(t),y(t))$.

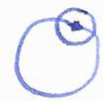
- any two continuous functions $x(t), y(t)$ determine a parameterised curve. or $t\mapsto z(t)$ where $z(t)=x(t)+iy(t)$
- the curve is closed if its first and last point are the same, i.e. $f(a)=f(b)$, otherwise we call it an arc 

Let E be a subset of $\mathbb{C}$. $E\subseteq\mathbb{C}$. E is path/arcwise connected if any pair of points $z,z'\in E$ can be connected by a path in E.



Recall a ~~an~~ open neighbourhood of $z\in\mathbb{C}$ is an open disc centered at z, i.e. $\{w\in\mathbb{C} \mid |z-w|<r\}$.

Defn A set $E\subseteq\mathbb{C}$ is open if every point $z\in E$ has ~~contains~~ an open nbd contained in E.

Example  ~~an open~~ neighbourhood is open. the closed disc  is not open

Defn A set $G\subseteq\mathbb{C}$ is a (complex) domain if it is path-connected and open.

Example  Non-example: $\mathbb{R}\subseteq\mathbb{C}$. Notation G^c = complement of G $=\mathbb{C}\setminus G$.

Defn Let $z\in G^c$. If there is a nbd of z contained in G^c, then z is an exterior pt for G. If every nbd of z contains points of both G, G^c, then z is a boundary point for G. Sometimes write ∂G for boundary of G.

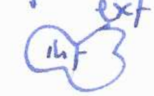
Example  ← boundary is circle $|z|=1$, exterior is $|z|>1$

$|z|<1$

- $\mathbb{C}\setminus[0,1]$ is a complex domain. boundary $=[0,1]$ exterior $=\emptyset$.

Defn given a domain G, let $\bar{G}=G\cup\partial G$, we call this a closed domain.

Defn Let $f: \underset{[a,b]}{I}\to\mathbb{C}$ be a closed curve, i.e. $f(a)=f(b)$. f is a Jordan curve if f is injective on $[a,b)$, i.e. $f(a)=f(b) \Rightarrow a=b$ on $[a,b)$.

Thm [Jordan curve theorem] A Jordan curve C cuts the plane into two open domains; bounded one is called the interior, other one called exterior 

Defn A domain G is simply connected if for every Jordan curve $C\subset G$, the interior of C is also contained in G. Otherwise it is not simply connected / multiply connected

Example  Non-example   $\mathbb{C}\setminus[0,1]$.

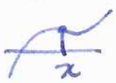
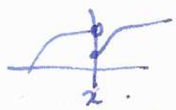
Defn for $\mathbb{C}\cup\{\infty\}$: $G \subseteq \mathbb{C}\cup\{\infty\}$ is simply connected, if for any Jordan curve C, $C \subset G$, either the exterior (containing ∞) or the interior is contained in G.

Example $\mathbb{C}\setminus[0,1]$ not simply connected in $\mathbb{C}$. is simply connected in $\mathbb{C}\cup\{\infty\}$

$\mathbb{C}\setminus\{0,1\}$ not simply connected in either.

Defn Let C_0 be a Jordan curve containing Jordan curves $C_1,\dots C_n$ with disjoint interiors. Then $C_0\setminus\{C_1,\dots C_n\}$ is $(n+1)$-connected  $\leftarrow$ triply connected.

Continuity Recall $f:\mathbb{R}\to\mathbb{R}$ is cts at x if $f(x)=\lim_{y\to x} f(y)$.

Example  Non-example 

Defn $f:G\to\mathbb{C}$ (G open domain) $z_0\in G$, $A\in\mathbb{C}$. $\lim_{z\to z_0} f(z)=A$ (or $f(z)\to A$ as $z\to z_0$).

if for all $\epsilon>0$, $\exists\,\delta>0$ s.t. $|f(z)-A|<\epsilon$ for all $|z-z_0|<\delta$.

Furthermore, if $f(z_0)=\lim_{z\to z_0} f(z)$ then we say f is cts at z_0. 

If f is continuous at all $z\in G$, we say f is cts on G. 

Example $f:\mathbb{C}\to\mathbb{C}$, $z\mapsto z^n$ $(n\in\mathbb{N})$ is cts on $\mathbb{C}$.

$$\underbrace{w-w_0}_{f(z)-f(z_0)} = z^n - z_0^n = (z-z_0)(z^{n-1}+z^{n-2}z_0+\dots+z_0^{n-1})$$

$$|f(z)-f(z_0)| \le |z-z_0|\,(r^{n-1}+r^{n-2}r_0+\dots+r_0^{n-1})$$

$r=|z|$, $r_0=|z_0|$.

$|z-z_0|<\delta \Rightarrow |z|<r_0+\delta$

so $|f(z)-f(z_0)| < \underbrace{|z-z_0|}_{\delta}\, n(r_0+\delta)^{n-1}$, so given ϵ, choose $\delta < \dfrac{\epsilon}{n(r_0+\delta)^{n-1}}$. $\square$.

Continuity for more general subsets, e.g. $\overline{G}$ or C.
— just restrict to points in subset; i.e. look at $|z-z_0|<\delta \cap \overline{G}$ / C etc…