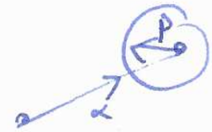
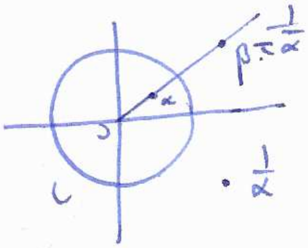
so $|\alpha-\beta| \geq |\beta| - |\alpha|$.

$\geq |\alpha| - |\beta|$

$\geq \big| |\alpha| - |\beta| \big|$.



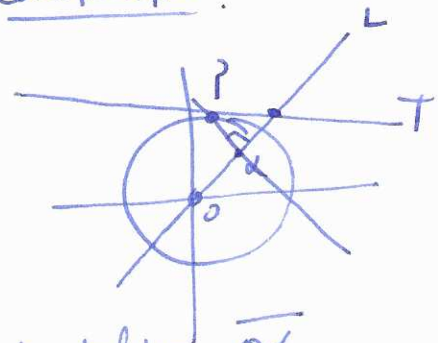
Inversion in a circle



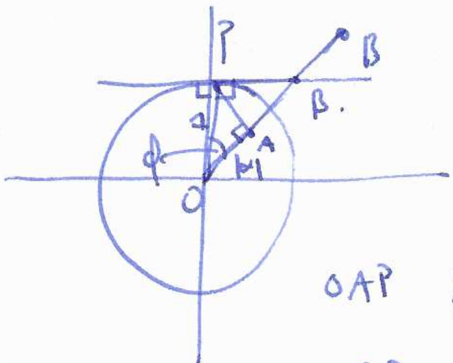
$|0\alpha||0\beta| = |\text{radius}(c)|^2$

$\alpha \cdot \frac{1}{\bar\alpha} = R^2$.

Construction.



1) construct line $\overline{O\alpha}$
2) find $\perp$ thru α find $P \in C \cap \perp$.
3) draw tangent T at $\cdot$ P
4) $\beta = T \cap L$.

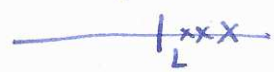


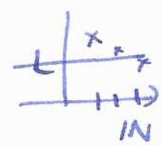
OAP $\quad \frac{|\alpha|}{1} = \cos\phi$.

OBP. $\quad \frac{1}{|\beta|} = \cos\phi \qquad \frac{|\alpha|}{1} = \frac{1}{|\beta|} \qquad |\alpha||\beta| = 1$.

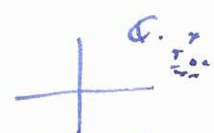
Limits

recall: an sequence of real numbers, then $\lim_{n\to\infty} a_n = L$ ($a_n \to L$) if $\forall \epsilon > 0\ \exists N$ s.t. $|L-a_n| < \epsilon$ for all $n \geq N$.



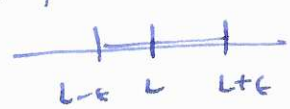


Defn z_n sequence of complex numbers, then $z_n \to w \in \mathbb{C}$ if $\forall \epsilon > 0\ \exists N$ s.t. $\forall n \geq N$ $|L-z_n| \leq \epsilon$ modulus.

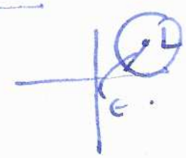


~~Remark in R~~ Recall ($\mathbb{R}$): Let $A \subseteq \mathbb{R}$ which is bounded above. Then A has a least upper bound (supremum A, $\sup A$) (observe: not true in $\mathbb{Q}$.)

Remark $\quad |L-x| \leq \epsilon$ defines an interval in $\mathbb{R}$:



$|L-z| \leq \epsilon$ defines a disc in $\mathbb{C}$.

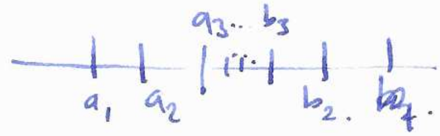


Thm (Nested intervals) I_n sequence of closed intervals in $\mathbb{R}$ s.t.

1) $I_{n+1} \subseteq I_n$ for all n
2) length$(I_n) \to 0$ as $n \to \infty$

$\Big\}$ Then $\bigcap I_n = \{x\}$ exists and is unique point.

Proof Let $I_n = [a_n, b_n]$ be the intervals



note $a_n \leq b_k$ for all n, so $(a_n)_n$ bounded.

Let a be the least upper bound of the (a_n), so in particular $a_n \leq a$ for all n.

claim: $a \in I_n$ for all n. Suppose not, then

$\Rightarrow b_n < a$ upper bound for (a_n) ✗.

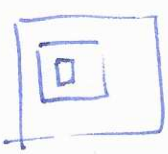
so $\bigcap I_n \ni a$. Suppose $\exists\, a \neq a' \in \bigcap I_n$, then length$(I_n) \geq |a - a'| \neq 0$ ✗.

so a is unique. $\square$.

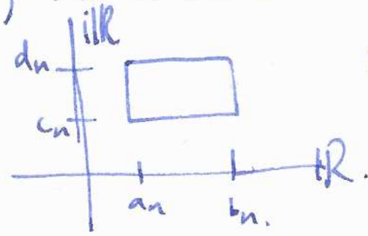
Thm (Nested rectangles) R_n nested rectangles w/ sides parallel to cardinal axes.

1) $R_{n+1} \subseteq R_n$
2) length diagonal$(R_n) \to 0$ as $n \to \infty$

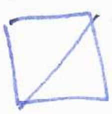
$\Big\}$ Then $\exists$ unique z s.t. $\bigcap R_n = \{z\}$.



Proof R_n



project R_n to $I_n \subseteq \mathbb{R}$. $\rightsquigarrow$ sequence of nested intervals.



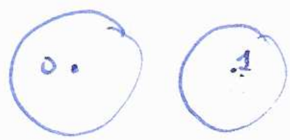
diagonal $\geq$ length of each side, implies length$(I_n) \to 0$, so previous result $\Rightarrow \exists$ unique $a \in \bigcap I_n$. Similarly project R_n to $J_n \subseteq i\mathbb{R}$.

$\rightsquigarrow$ sequence of nested intervals $\exists$ unique $b \in \bigcap J_n \Rightarrow a + bi$ unique point in $\bigcap R_n$ $\square$

Limits vs limit point $\boxed{\text{sequence } (z_n) \text{ vs set } \{z_n\}.}$

Defn A complex number z is ~~the limit~~ a limit point of the sequence $(z_n)_{n \in \mathbb{N}}$ if for all $\epsilon > 0$ $|z_n - z| \leq \epsilon$ holds for infinitely many n.

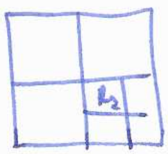
Example
- $(0, 1, 0, 1, 0, 1, 0, \ldots)$ limit does not exist, but both 0 and 1 are limit points.
- $\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{1}, \frac{4}{5}, \ldots\right)$
- $(0, 1, 0, 2, 0, 3, 0, 4, \ldots)$
- $(0, 1, 2, 3, 4, \ldots)$



Def^n the sequence (z_n) is **bounded** if $\exists M \in \mathbb{R}$ s.t. $|z_n| \le M$ for all n.

Thm (Bolzano-Weierstrass) Every bounded sequence (z_n) in $\mathbb{C}$ has at least one limit point.

Proof Every ~~elem~~ z_n lies in some rectangle R_1 w/ sides parallel to axes.



R_1

cut R_1 into 4 congruent rectangles. At least one contains infinitely many points R_2 divide this one ... this gives an infinite sequence of nested rectangles, so has a unique intersection point z. Any disc about z contains some rectangle R_n, so contains ∞-many points so z is a limit point of (z_n). □.

Recall $\lim_{n\to\infty} (z_n) = z$ if $\forall \epsilon > 0 \; \exists N$ s.t. $|z_n - z| \le \epsilon \; \forall n \ge N$.

Thm/Fact If $\lim z_n = z$ and $\lim z_n' = z'$

then $\lim_{n\to\infty} (z_n + z_n') = z + z'$ $\quad$ $\lim_{n\to\infty} (z_n z_n') = zz'$

$\lim_{n\to\infty} (z_n - z_n') = z - z'$ $\quad$ $\lim_{n\to\infty} \left(\frac{z_n}{z_n'}\right) = \frac{z}{z'}$ as long as $z' \ne 0$ □.

Thm (Cauchy convergence thm) z_n converges iff for any $\epsilon > 0 \; \exists N$ s.t.

$|z_m - z_n| < \epsilon \quad \forall m, n > N$.

Warning: what goes wrong for $z_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

Proof $\Rightarrow$ suppose $\lim_{n\to\infty} (z_n) = z$. Then given $\epsilon/2$ there is N s.t. $|z_n - z| \le \frac{\epsilon}{2}$ for all $n \ge N$. Then $|z_n - z_m| \le |(z_n - z) + (z - z_m)| \le |z_n - z| + |z_m - z| \le \epsilon$.

$\Leftarrow$ Choose $\epsilon = 1$ say, ~~then $\exists N_0$~~ with corresponding N. Choose z_N. then $|z_N - z_m| < 1$ for all $m \ge N$ $\Rightarrow$ sequence is bounded, so Bolzano-Weierstrass $\Rightarrow$ there is a limit point z say. Now consider $\epsilon/2$, and corresponding $N' = N(\epsilon/2)$, so $|z_n - z| \le \frac{\epsilon}{2}$ and $|z_n - z_m| \le \epsilon/2$ for all $n \ge N'$. Then $|z_n - z| = |(z_n - z_m) + (z_m - z)| \le \underbrace{|z_n - z_m|}_{\le \epsilon/2 \text{ Cauchy criterion}} + \underbrace{|z_m - z|}_{< \frac{\epsilon}{2} \; z \text{ is limit point}} \le \epsilon$ as required □