MTH 431 Complex Analysis

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office hours M 2:30 - 4:25 W 3:35 - 4:25

- students w/ disabilities

Text: Complex Analysis w/ application, Silverman.

Complex numbers

motivation counting numbers 1, 2, 3, 4, 5, ... $\mathbb{N}$ addition $a+b$ multiplication ab

solving equations: · if I give you 2 apples and still have 3, how many did I start with?

$x-2=3 \rightsquigarrow x=5$

can't solve: · if I have \$10 and give you \$20 how much do I have? $10-20=?=-10$

integers, includes negative numbers (and zero).

$\ldots -2, -1, 0, 1, 2, 3, \ldots$ $\mathbb{Z}$ addition/subtraction $a+b$ $a-b$. multiplication

can now solve $x+3=2 \rightsquigarrow x=-1$.

can't solve: · 2 of us have 3 apples, how many apples each?

$2x=3. \rightsquigarrow x=\frac{3}{2}$ ← fractions

$\mathbb{Q} = \{ \frac{a}{b} \mid a,b \in \mathbb{N},\ b \neq 0 \}$. addition/subtraction multiplication/division. $\frac{a}{b} \in \mathbb{Q}\ (b \neq 0)$

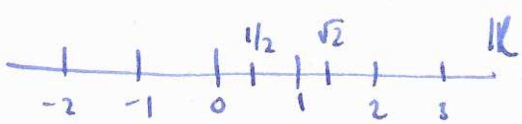
can't solve: · what is the perimeter of a circle? $2\pi r$
· what is the length of the diagonal of a square? $\sqrt{2}$

Fact: $\pi, \sqrt{2} \notin \mathbb{Q}$, but they are real numbers. $\mathbb{R}$.

$\mathbb{R}$ = "numbers w/ infinite decimal expansions, not nec. repeating".

$\pi = 3.141592654\ldots$ $1/3 = 0.333\ldots$ $1 = 1.000\ldots$

number line:



$x \mapsto x+1$ translation
$x \mapsto x+a$
$x \mapsto 2x$ expansion
$x \mapsto x/2$ contraction $x \mapsto ax$.
$x \mapsto -x$ reflection

can solve $x^2-2=0$ can't solve: $x^2+1=0$.

solution: create a new number i, s.t. $i^2=-1$, then $(i)^2+1=-1+1=0$ ✓.

want: i satisfies same number rules as $\mathbb{R}$: $1+i$, $\frac{i}{4}$ gives numbers of the form $a+bi$ $\underset{\text{complex numbers}}{\mathbb{C}} = \{a+bi \mid a,b \in \mathbb{R}\}$.

<u>Fact</u>: this works. <u>Amazing fact</u>: all polynomials can be solved over $\mathbb{C}$.

e.g. $x^3+1=0$ $x^4+1=0$ $x^5+3x-1=0$.

↑

<u>check</u>

addition: $a+bi+c+di = (a+c)+(b+d)i$

subtraction: $a+bi-(c+di) = (a-c)+(b-d)i$

multiplication: $(a+bi)(c+di) = ac+adi+bic+\underbrace{bidi}_{bd\cdot i^2=-bd.}$

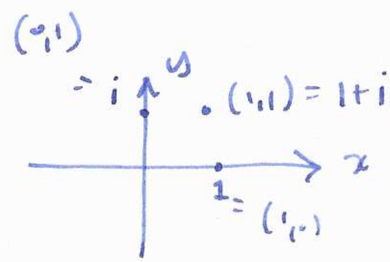
$= ac-bd+(ad+bc)i$

division: $\frac{a+bi}{c+di}$? $\frac{a+bi}{c+di}\frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-\cancel{cdi}+\cancel{dic}+d^2} = \frac{ac+bd+(bc-ad)i}{c^2+d^2}$.

↑

$c+di \neq 0$!

$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$



<u>Notation</u> $a+0i = a$ real number, so $\mathbb{R} \subseteq \mathbb{C} \cong \mathbb{R}^2$

<u>Exercise</u>: check $\mathbb{C}$ is a field.

1) addition commutative $\alpha+\beta=\beta+\alpha$

associative $(\alpha+\beta)+\gamma= \alpha+(\beta+\gamma)$

inverse $\alpha+(-\alpha)=0 \leftarrow$ additive identity

<u>Fact</u>. not fields: $\mathbb{R}^n$, matrices, functions?

2) multiplication commutative $\alpha\beta=\beta\alpha$

associative $(\alpha\beta)\gamma= \alpha(\beta\gamma)$

inverse $\alpha\cdot\frac{1}{\alpha}=1 \leftarrow$ multiplicative identity

3) distribution: $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$.

<u>Notation</u> $\underset{\text{real part}}{a}+\underset{\text{imaginary part}}{b}i$ $\mathrm{Re}(a+bi)=a$ $\mathrm{Im}(a+bi)=b$.

<u>Fact</u> two complex numbers are equal if they have the same real and imaginary parts: $a+bi=c+di \Leftrightarrow \begin{matrix} a=c \\ b=d \end{matrix}$.

Complex conjugate $\alpha = a+bi$ define $\bar{\alpha} = a-bi$

observation: $\alpha\bar{\alpha} = (a+bi)(a-bi) = a^2+b^2 \in \mathbb{R}$.

so $\bar{\alpha} = \dfrac{a^2+b^2}{\alpha}$ so $\dfrac{1}{\alpha} = \dfrac{\bar{\alpha}}{a^2+b^2}$

Fact If $\alpha\beta = 0$ then at least one of α, β is equal to 0
(no zero divisors)

Proof suppose $\alpha \neq 0$ then $\frac{1}{\alpha}\alpha\beta = 0 \Rightarrow 1\beta = 0 \Rightarrow \beta = 0$. $\square$.

Remark matrices have zero divisors: $AB = 0 \not\Rightarrow$ at least one of A, B zero

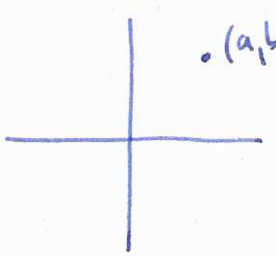
Observation $\overline{\alpha+\beta} = \bar{\alpha} + \bar{\beta}$ ① $\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}$. $\alpha = \beta \Leftrightarrow \bar{\alpha} = \bar{\beta}$.

check: ① $\overline{(a+bi)+(c+di)}$ $\quad \overline{a+bi} + \overline{c+di}$

$= \overline{a+c+(b+d)i}$ $\quad a-bi+c-di$

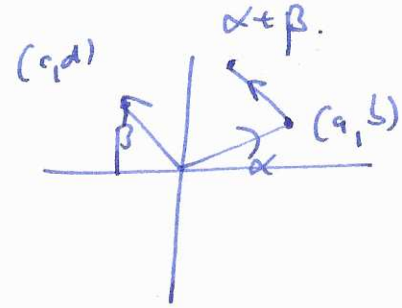
$= a+c-(b+d)i$ $\quad a+c-(b+d)i$ $\checkmark$.

Complex plane $\mathbb{C} \cong \mathbb{R}^2$

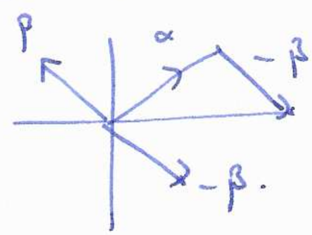


addition $\Leftrightarrow$ vector addition

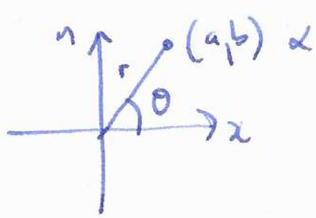
$(a+bi)+(c+di)$



$\alpha - \beta$:



Polar coordinates / modulus, argument



$|\alpha| = \sqrt{\bar{\alpha}\alpha} = \sqrt{a^2+b^2} = r$

modulus r

argument θ $\quad \tan\theta = b/a$.

Remark
- $\arg(0)$ undefined!
- any 2π to θ doesn't change complex number!

$a = r\cos\theta$
$b = r\sin\theta$ so $\alpha = a+bi = r\cos\theta + r\sin\theta\, i = r(\cos\theta + i\sin\theta)$ $re^{i\theta}$.

products: $\alpha = a+bi = r(\cos\theta + i\sin\theta)$ $\alpha\beta = rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$.

$\beta = c+di = s(\cos\phi + i\sin\phi)$ $\quad = rs\left(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\right)$

$= rs\left(\cos(\theta+\phi) + i\sin(\theta+\phi)\right)$

④

so $|\alpha\beta| = |\alpha||\beta|$ $\quad$ $\arg(\alpha\beta) = \arg(\alpha) + \arg(\beta)$ $\quad$ $\sim$ $re^{i\theta} \cdot se^{i\phi}$

multiply moduli $\qquad$ add arguments. $\qquad$ $= rs\, e^{i(\theta+\phi)}$.

Action on $\mathbb{C}$:

$z \mapsto z + \alpha$ translation

$z \mapsto \alpha z$ rotation and scaling by $|\alpha|$.
(by $\arg(\alpha)$)

Exercise $\quad |\alpha_1 \alpha_2 \dots \alpha_n| = |\alpha_1||\alpha_2| \cdots |\alpha_n|$

$\arg(\alpha_1 \alpha_2 \dots \alpha_n) = \arg(\alpha_1) + \arg(\alpha_2) + \dots + \arg(\alpha_n)$

special case: $\quad |\alpha^n| = |\alpha|^n \qquad \arg(\alpha^n) = n \arg(\alpha)$

in polars: $\quad (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i \sin n\theta)$ (De Moivre's Thm).

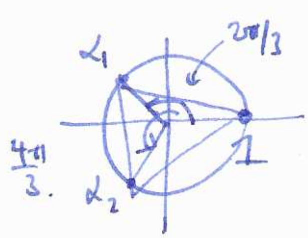
Complex roots $\quad \sqrt[n]{\alpha} \leftarrow$ any complex number β s.t. $\beta^n = \alpha$.

modulus $|\sqrt[n]{\alpha}| = \sqrt[n]{|\alpha|}$. $\quad$ argument: $\arg(\sqrt[n]{\alpha}) = \dfrac{\arg(\alpha)}{n}$. ⊛

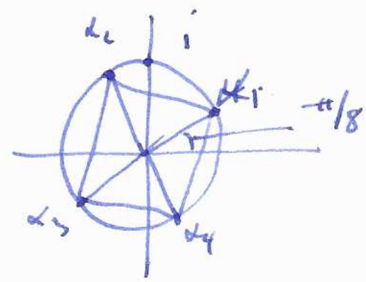
almost: remember can replace θ by $\theta + 2\pi k$.

so $\arg(\sqrt[n]{\alpha}) = \dfrac{\arg(\alpha) + 2\pi k}{n}$

Example $\quad \sqrt[3]{1}$



$\sqrt[4]{i}$

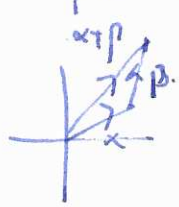


Division: $\quad z \mapsto \alpha z \qquad z \mapsto re^{i\theta} z$.

$z \mapsto \dfrac{z}{\alpha} \qquad z \mapsto \dfrac{1}{r} e^{-i\theta} z$.

check. $\quad \alpha \cdot \frac{1}{\alpha} = 1. \quad re^{i\theta} \cdot \frac{1}{r} e^{-i\theta} = 1 \cdot e^0 = 1$.

Triangle inequality



$|\alpha + \beta| \leq |\alpha| + |\beta|$

$|\alpha + \beta - \alpha| = |\beta| \leq |\alpha| + |\beta - \alpha|$

$\quad = |\alpha - \beta|$.