Math 431 Complex Analysis Spring 21 Sample Final Questions

(1) Find all cube roots of $-1-i$.

(2) Find all values of $(-i)^{-i}$.

(3) Write down a Möbius map f which takes the unit disc to the upper half plane. The map $z \mapsto z+1$ preserves the upper half space, what is the corresponding map on the unit disc? Draw the invariant circles for the action of $z \mapsto z+1$ on the plane, and the corresponding map on the disc.

(4) The map $z \mapsto \overline{z}$ acts on the complex plane as reflection in the real axis. Find a map which corresponds to reflection in the imaginary axis. Show that the product of two reflections is either a rotation or a translation.

(5) Use the Cauchy Riemann equations to show that if $f(z)$ is complex analytic / differentiable in a domain G in $\mathbb{C}$, and takes values in $\mathbb{R} \subset \mathbb{C}$, then f is constant.

(6) Let $u(x,y) = ax^2 + bxy + cy^2$ be a real quadratic function of two variables, where a, b and c are real constants. Show that u is harmonic if and only if $a = -c$. If u is harmonic then show that it is the real part of a function of the form $f(z) = Az^2$ for some $A \in \mathbb{C}$. Give a formula for A in terms of a, b and c.

(7) Let $z = x + iy$ and show that $|\sin z|^2 = \sin^2 x + \sinh^2 y = \cosh^2 y - \cos^2 x$.

(8) Does $\log(z^2) = 2\log(z)$?

(9) Find the radius of convergence of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{2^{\sqrt{n}}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2z}{n}\right)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} z^{n^n}$

(10) Find power series centered at zero for the following functions.

(a) $f(z) = \dfrac{z}{1+\frac{1}{z}}$
(b) $f(z) = \sin(z^2)$
(c) $f(z) = \sin^2(z)$

(11) Find the following Laurent series, and specify where they converge.

(a) $f(z) = \dfrac{1}{z(z-2)^2}$, centered at 2.
(b) $f(z) = \dfrac{z-2}{z+1}$, centered at -1.
(c) $f(z) = \dfrac{1}{(1-z)(z+2)}$, for $1 < |z| < 2$.

(12) Find the power series for e^z centered at $z = -1$. Find $\displaystyle\int_C \frac{e^z}{(1+z)^{23}}$, where C is the circle of radius 2, centered at -2.

(13) Compute $\displaystyle\int_C \frac{e^{z^2}}{z^3}\, dz$, where C is the unit circle around the origin, oriented anticlockwise.

(14) Compute $\displaystyle\int_C \frac{\sin(z)}{(z+3)(z^2+1)}\, dz$, where C is the unit circle around the origin, oriented anticlockwise.

(15) How many zeros does $f(z) = z^7 - 4z^3 + 1$ have in the region $1 \leq |z| \leq 2$?

(16) Find $\displaystyle\int_C \frac{e^z}{\sin z}\, dz$, where C is the circle of radius 3 centered at 2.

(17) Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx$ by doing a contour integral.

(18) Find $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^4}\, dx$ by doing a contour integral.