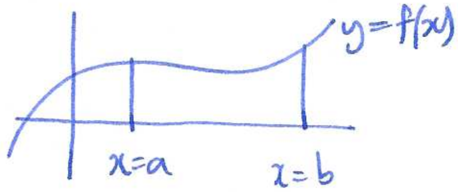
Example $c(t) = (\cos t, \sin 2t)$

Q: draw this!

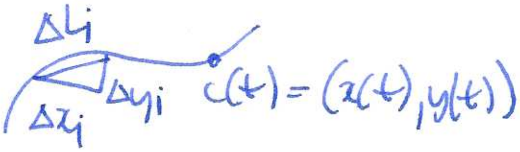
$$\left.\begin{aligned} x(t) &= \cos t & y(t) &= \sin 2t \\ \frac{dx}{dt} &= -\sin t & \frac{dy}{dt} &= 2\cos 2t \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{2\cos 2t}{-\sin t}$$

§11.2 Arc length + speed

recall



$$\text{arc length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

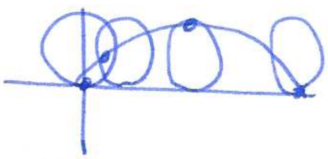


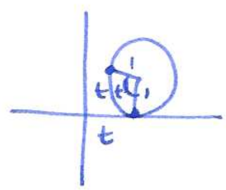
$$\text{arc length} \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$$

$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

Example cycloid





$$x(t) = t - \sin t \qquad \frac{dx}{dt} = 1-\cos t$$

$$y(t) = 1-\cos t \qquad \frac{dy}{dt} = \sin t$$

arc length: $\displaystyle\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$

$$= \int_0^{2\pi} \sqrt{2-2\cos t}\, dt$$

recall $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

$$= \int_0^{2\pi} \sqrt{4\sin^2 \tfrac{t}{2}}\, dt$$

$2\sin^2\theta = 1 - \cos 2\theta$

$2\sin^2\left(\frac{t}{2}\right) = 1 - \cos t$

$$= \int_0^{2\pi} 2\left|\sin\tfrac{t}{2}\right| dt = 4\int_0^{\pi} \sin\tfrac{t}{2}\, dt = 4\left[-2\cos\tfrac{t}{2}\right]_0^{\pi} = 8(0+1) = 8.$$

speed

$c(t) = (x(t), y(t))$

$$\text{speed} = \frac{d}{dt}(\text{arc length}) = \frac{d}{dt}\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \text{length of velocity vector}$$

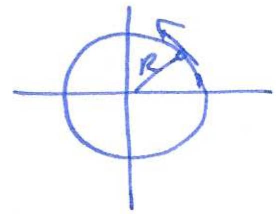
Example ① $c(t) = (t, t^2)$ $\quad x(t) = t \quad \frac{dx}{dt} = 1$

$y(t) = t^2 \quad \frac{dy}{dt} = 2t \quad$ speed $= s'(t) = \sqrt{1+4t^2}$.

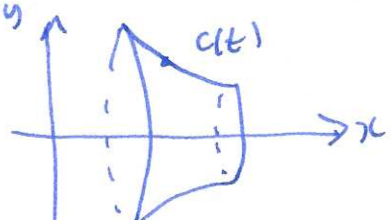
② $c(t) = (R\cos(\omega t), R\sin(\omega t))$

$c'(t) = (-\omega R\sin(\omega t), \omega R\cos(\omega t))$

$\|c'(t)\| = \sqrt{R^2\omega^2\sin^2\omega t + R^2\omega^2\cos^2(\omega t)} = R|\omega|$



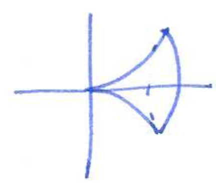
__surface area for parameterized curves__



$$\text{surface area} = 2\pi\int_a^b y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

__example__ $\quad c(t) = (t, t^3)$

$c'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1, 3t^2)$



$$\text{surface area} = 2\pi\int_0^1 t^3\sqrt{1^2+(3t^2)^2}\, dt = 2\pi\int_0^1 t^3\sqrt{1+9t^4}\, dt$$

$u = 1+9t^4$

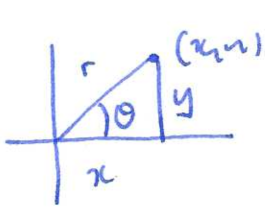
$\frac{du}{dt} = 36t^3$

$$2\pi\int_0^{10} t^3\sqrt{u}\,\frac{dt}{du}\, du = 2\pi\int_0^{10} t^3\sqrt{u}\,\frac{1}{36t^3}\, du = \frac{\pi}{18}\int_1^{10}\sqrt{u}\, du$$

$$= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\pi}{27}\left(10^{3/2} - 1\right) \approx 3.56$$

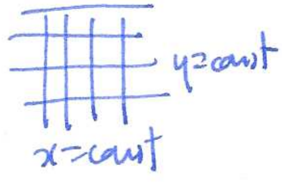
Example $\quad c(t) = (t - \tanh(t), \operatorname{sech}(t))$

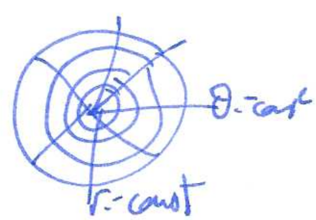
__§11.3 Polar coordinates__



$x = r\cos\theta \quad \tan\theta = y/x$

$y = r\sin\theta \quad r^2 = x^2 + y^2$





__Equations in polar coordinates__

circle of radius R: cartesian $\quad x^2+y^2 = R^2$

polar $\quad r = R$