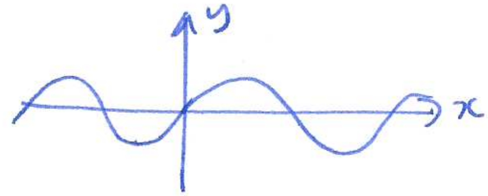
so $T_n^{(k)}(a) = k!\,a_k = f^{(k)}(a)$ so $a_k = \frac{1}{k!} f^{(k)}(a)$

so $T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

<u>Example</u> 1 $y = \sin x$ at $x = 0$



$f(x) = \sin x$ $f(0) = 0$

$f'(x) = \cos x$ $f'(0) = 1$

$f''(x) = -\sin x$ $f''(0) = 0$

$f^{(3)}(x) = -\cos x$ $f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$ $f^{(4)}(0) = 0$

so $T_4(x) = 0 + 1.x + \frac{0}{2!}.x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}.x^4$

$= x - \frac{x^3}{3!}$

2 $y = \cos x$ at $x = 0$

$f(x) = \cos x$ $f(0) = 1$

$f'(x) = -\sin x$ $f'(0) = 0$

$f''(x) = -\cos x$ $f''(0) = -1$

$f^{(3)}(x) = \sin x$ $f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x$ $f^{(4)}(0) = 1$

so $T_4(x) = 1 + 0.x + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + 1\frac{x^4}{4!}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

3 $y = e^x$ at $x = 0$

$f(x) = e^x$ $f(0) = 1$

$f'(x) = e^x$ $f'(0) = 1$

$f''(x) = e^x$ $f''(0) = 1$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

4 $y = \ln(x)$ at $x = 1$

$f^{(k)}(x) = (-1)^{k+1}(k-1)!\,x^{-k}$

$f^{(k)}(1) = (-1)^{k+1}(k-1)!$

$f(x) = \ln(x)$ $f(1) = 0$

$f'(x) = 1/x = x^{-1}$ $f'(1) = 1$

$f''(x) = -x^{-2}$ $f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $f^{(3)}(1) = 2$

$f^{(4)}(x) = -6x^{-4}$. $f^{(4)}(1) = -3!$

$T_n(x) = 0 + (x-1) + (-1)\frac{(x-1)}{2!} + \frac{2!}{3!}(x-1)^2 + \dots$

$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$

Q: how good are the Taylor polynomial approximations?

Thm (error bound) $f(x)$ function, $f^{(n)}(x)$ exists and is continuous.

Let K be an upper bound for $|f^{(n+1)}(u)|$ for all u in $[a, x]$.

then $\displaystyle |T_n(x) - f(x)| \leq \frac{K|x-a|^{n+1}}{(n+1)!}$

Example $f(x) = e^x$ find $T_4(x)$ at $x = 0$, find an error bound for $T_4(1)$

$$\left.\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ \vdots & \\ f^{(4)}(x) = e^x & f^{(4)}(0) = 1 \end{array}\right\} \quad \begin{array}{l} T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \\ T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.708\overline{33} \end{array}$$

error bound: need to find K s.t. $|f^{(n+1)}(x)| \leq K$ for all $x \in [0,1]$

i.e. want $|e^u| \leq K$ for $u \in [0,1]$, can choose $K = 3$.

so $\displaystyle |e - T_4(1)| \leq \frac{3 \cdot 1^{n+1}}{(n+1)!} = \frac{3}{120}$ $\left[\begin{array}{l} \text{quick upper bound for } e: \\ e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \\ \quad \leq 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3. \end{array} \right]$.

§10.8 Taylor series

suppose $f(x)$ has a power series expansion at $x = a$, i.e.

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

Q: how do we find the a_i?

A: $f(a) = a_0$

$$f'(x) = a_1 + 2a_2(x-a)^2 + 3a_3(x-a)^2 + \cdots$$

$$f'(a) = a_1$$

$$f''(x) = 2a_2 + 3.2a_3(x-a) + 4.3a_4(x-a)^2 + \cdots$$

$$f''(a) = 2a_2$$

$\vdots$ $f^{(k)}(x) = k!a_k$ $\qquad f^{(k)}(a) = k!a_k.$ $\qquad \boxed{a_k = \frac{f^{(k)}(a)}{k!}}$

Examples $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$

Thm If $f(x)$ is equal to a power series centered at $x=c$, with radius of convergence $R>0$, then $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, where $a_n = \frac{f^{(n)}(c)}{n!}$

Useful facts where the Taylor series converges, we can:

- differentiate
- integrate
- multiply (xe^x)
- substitute (e^{x^2})

Examples xe^x, e^{-x^2}, $e^x \sin x$

Fact works for complex numbers $x+iy$ $\qquad e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$

$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots$

$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$

$= \cos\theta + i\sin\theta$

Application: double angle formula

$$\left.\begin{aligned} e^{2i\theta} &= \cos 2\theta + i\sin 2\theta \\ (e^{i\theta})^2 &= (\cos\theta + i\sin\theta)^2 \end{aligned}\right\} \text{ also triple angle formula!}$$

How to make l'Hôpital's rule problems:

$$\left.\begin{aligned} \sin(2x) &\approx 2x - \frac{(2x)^3}{3!} + o(x^5) \\ e^{3x} &\approx 1 + 3x + o(x^2) \end{aligned}\right\} \qquad \frac{\sin(2x)}{e^{3x}-1} \approx \frac{2x}{3x} = \frac{2}{3}.$$

cheap error bounds for e^x: $(x<1)$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \underbrace{\frac{x^{n+1}}{(n+1)!} + \cdots}_{\leq\, x^{n+1} + x^{n+2} + \cdots \,=\, \frac{x^{n+1}}{1-x}}$$

so $e^{1/2} = 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \cdots + \frac{(1/2)^n}{n!}$ with error $\leq \frac{(1/2)^{n+1}}{1-1/2} = \frac{1}{2^n}$