so $a_n \not\to 0 \Rightarrow \sum a_n$ diverges $\square$.

Example ① show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

② show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{n^3/3^n} = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$

Bad example $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} = 1.$

Thm **Root test** (a_n) sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists

① if $L < 1$ then $\sum a_n$ converges absolutely

② if $L > 1$ then $\sum a_n$ diverges

③ if $L = 1$ no information.

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

§10.6 Power series

Defn A power series centered at $x=a$ is an infinite sum of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

note: if this converges, it gives a function of x.

the series always converges if $x=a$! $F(a) = a_0$

Thm **Radius of convergence** Let $F(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then

① $F(x)$ converges only for $x=a$ $(R=0)$

or ② $F(x)$ converges for all $x \in \mathbb{R}$ $(R=\infty$

or ③ there is an $R>0$ s.t. $F(x)$ converges for all $|x-a| < R$, and

diverges for all $|x-a|>R$. It may or may not converge at $a+R, a-R$.
R is called the radius of convergence.

Example ① for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{2} = \frac{1}{2}|x|$.

so converges for $\frac{1}{2}|x|<1 \iff |x|<2$ (so $R=2$) check endpoints!

② $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(x-5)^n$ ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{x^{n+1}} \cdot \frac{x^n}{(x-5)^n} \right|$

$= \lim_{n\to\infty} \left|(x-5)\frac{n}{n+1}\right| = |x-5|<1$, so converges for $|x-5|<1$, $R=1$

Examples $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n!\,x^n$

Thm Suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence $R>0$,
then $F(x)$ is differentiable on $(a-R, a+R)$ and furthermore:

$$\left.\begin{aligned} F'(x) &= \sum_{n=1}^{\infty} a_n\, n (x-a)^{n-1} \\ \int F(x)\,dx &= \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + c \end{aligned}\right\}$$ with the same radius of convergence.

Example ① $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ radius of convergence $R=1$

so $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$

and $\int \frac{1}{1-x}\,dx = -\ln|1-x| = c+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$

$$\left.\begin{aligned} &② \; 1-x^2+x^4-x^6+\cdots = \frac{1}{1+x^2} \\ &\text{so } \tan^{-1}(x) = c+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \end{aligned}\right\}$$ Q: radius of convergence?

③ <u>Fact</u> $y = e^x$ is the (unique up to scalar multiple) solution to

$\frac{dy}{dx} = y \quad (f'(x) = f(x)) \qquad \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

so $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ radius of convergence $R = \infty$.

How to find a power series solution to a differential equation:

try: $y = a_0 + a_1 x + a_2 x^2 + \cdots$

<u>Example</u> $y' = y$: $a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$

$a_1 = a_0$

$2a_2 = a_1 = a_0 \Rightarrow a_2 = a_0/2$ so $y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

$3a_3 = a_2 \Rightarrow a_3 = a_0/3!$

<u>Example</u> $y'' + y' + y = 0$ with $y(0) = 1$ $y'(0) = 1$

Look for solution $y = a_0 + a_1 x + a_2 x^2 + \cdots$ $\quad y(0) = 1 \Rightarrow a_0 = 1$

$y'(0) = 1 \Rightarrow a_1 = 1$

$y = 1 + x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = 1 + 2a_2 + 3a_3 x^2 + \cdots$

$y'' = 2 + 6a_3 + \cdots$

$y'' + y + y' = (1 + 1 + 2a_2) + x(1 + 2a_2 + 6a_3) + x^2(\text{—}) + \cdots = 0$

$\Rightarrow a_2 = -1 \quad a_3 = -1/6$, etc.

<u>§10.7 Taylor series</u>

suppose $f(x)$ has a power series expansion at $x = c$, i.e.

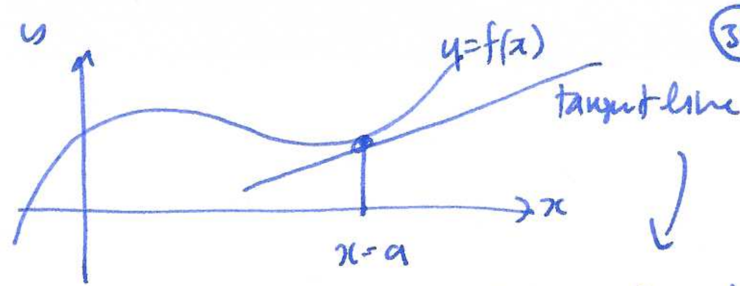
$f(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots$

Q: how do we find the a_i?

A: $f(c) = a_0$

§10.7 Taylor polynomials

Approximating functions at $x=a$:



0-th approx : $f(x) \approx f(a)$

1-st order approx : (straight line) $f(x) \approx f(a) + f'(a)(x-a)$

$y - y_0 = m(x - x_0)$

$y - f(a) = f'(a)(x-a)$

2nd order approx : (quadratic) $\}$ how do we find these?
3rd order approx : (cubic)

tangent line : unique line with same value as $f(a)$ at $x=a$
same derivative $f'(a)$ at $x=a$

quadratic : same value at $x=a$ $f(a)$
same derivative at $x=a$ $f'(a)$
same 2nd derivative at $x=a$ $f''(a)$

cubic : same value, 1st derivative, 2nd derivative, 3rd derivative

etc.

quadratic $T_2(x) = ax^2 + bx + c$

better : $T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$

deg 0 : $T_2(a) = a_0 \qquad = f(a)$ (same value).

deg 1 : $T_2'(x) = a_1 + 2a_2(x-a)$

$T_2'(a) = a_1 \qquad = f'(a)$ (same 1st derivative)

deg 2 : $T_2''(x) = 2a_2$

$T_2''(a) = 2a_2 \qquad = f''(a)$ (same 2nd derivative)

so $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

in general $T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

differentiate k-times : $T_n^{(k)}(x) = k!\,a_k +$ (stuff with $(x-a)$ factors)