Proof (sketch) case $L > 0$, then $\frac{a_n}{b_n} \to L$, so $0 < \frac{a_n}{b_n} < R$ for some $L < R$

$$0 < a_n < R b_n$$

comparison test: $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

similarly $\frac{b_n}{a_n} \to \frac{1}{L}$, so $0 < \frac{b_n}{a_n} < R'$ for some $\frac{1}{L} < R'$, $0 < b_n < R' a_n$

so $\sum b_n$ converges by comparison test. (if $L = 0$ only get one direction) $\square$.

Example show $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges. (for large n, $a_n \sim \frac{1}{n^2}$)

compare with $b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n^2 / n^4 - n - 1}{1/n^2} = \frac{n^4}{n^4 - n - 1} = 1$

so $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges, by limit comparison test.

Example does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2 - 9}}$ converge? compare with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2-9}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2 - 9}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 - 9/n^2}} = 1$$

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-9}}$ diverges, by limit comparison test.

§10.4 Absolute and conditional convergence

Q: what about $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$? (*)

Def^{n} A series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

(*) is absolutely convergent.

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ <u>not</u> absolutely convergent

Thm Absolute convergence $\Rightarrow$ convergence.

<u>Proof</u> $0 \le a_n + |a_n| \le 2|a_n|$

$\sum_{n=1}^{\infty} 2|a_n| = 2\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n + |a_n|$ converges, by comparison test.

then $\sum_{n=1}^{\infty} (a_n + |a_n|) - |a_n| = \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$

converges $\Leftarrow$ converges converges

$= \sum_{n=1}^{\infty} a_n$, so this converges. $\square$.

<u>Q</u>: what about $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

<u>Defn</u> $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u> if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

<u>Thm</u> (Alternating series test) Let a_n be a positive, decreasing sequence, with $a_n \to 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges (to s say).

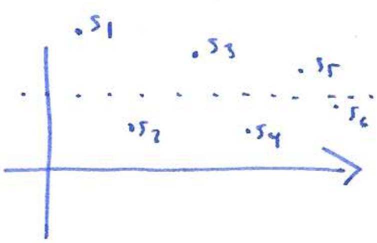
Furthermore $0 \le s \le a_1$ and $s_{2n} \le s \le s_{2n+1}$ for all n.

<u>Proof</u>: even partial sums: $s_{2n} = \underbrace{a_1 - a_2}_{\ge 0} + \underbrace{a_3 - a_4}_{\ge 0} + \dots + \underbrace{a_{2n-1} - a_{2n}}_{\ge 0}$

positive increasing sequence $\uparrow$

odd partial sums: $s_{2n+1} = a_1 - \underbrace{(a_2 - a_3)}_{\ge 0} - \underbrace{(a_4 - a_5)}_{\ge 0} - \dots - \underbrace{(a_{2n} - a_{2n+1})}_{\ge 0}$

$\downarrow$ decreasing sequence



furthermore $s_{2n} = a_1 - (a_2 - a_3) - \dots - a_{2n}$

so $s_{2n} \le a_1$ for all n

so s_{2n} is an increasing sequence, bounded above by a_1 so $\lim_{n\to\infty} s_{2n}$ exists.

similarly $\lim_{n\to\infty} s_{2n+1}$ exists.

finally: $\lim_{n\to\infty} s_{2n} - s_{2n+1} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} -a_{2n+1} = 0$

so $\lim_{n\to\infty} s_n$ exists. $\square$

<u>Example</u> show $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$ converges (alternating harmonic series)

use alternating series test. $a_n = \frac{1}{n}$ a_n positive, decreasing, $a_n \to 0$

so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges $\square$.

so $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$ is conditionally convergent.

§10.5 Ratio and root tests

<u>fact</u>: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$ Q: how do we show this converges?

(e.g: use comparison test $n! = 1 \cdot 2 \cdot 3 \cdot \dots (n-2)(n-1)n > (n-1)^2$

so $\frac{1}{n!} < \frac{1}{(n-1)^2}$)

<u>Thm</u> <u>Ratio test</u> (a_n) sequence, and suppose $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists

then ① if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

② if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

③ if $\rho = 1$ no information.

<u>Proof</u> if $\rho < 1$, there is a number $\rho < r < 1$, and a number N s.t

$\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq N$, so $|a_{N+1}| < r|a_N|$

$|a_{N+1}| < r|a_{N+1}| < r^2|a_N|$ etc

so $\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| r^n \leq \frac{|a_N|}{1-r}$ so converges by comparison with geometric series.

if $\rho > 1$, then there is ~~a~~ $\rho > r > 1$ and N s.t. $\frac{|a_{n+1}|}{|a_n|} > r$ for all $n \geq N$