General case

$$c + cr + cr^2 + \cdots = \sum_{n=0}^{\infty} cr^n$$

$$S_N = c + cr + cr^2 + \cdots + cr^N$$

$$rS_N = cr + cr^2 + \cdots + \cancel{cr^N} + cr^{N+1}$$

$$S_N - rS_N = c - cr^{N+1}$$

$$S_N(1-r) = c(1-r^{N+1})$$

$$S_N = \frac{c(1-r^{N+1})}{1-r}$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{c(1-r^{N+1})}{1-r}$$

$$\lim_{N\to\infty} = \frac{c}{1-r} \quad \text{if } |r|<1$$

otherwise does not converge.

special series (telescoping)

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

so $S_N = 1 - \frac{1}{N+1}$ so $\lim_{N\to\infty} S_N = 1$.

Example (harmonic series) $\quad 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}.} + \cdots$

diverges!

Q: when does a series converge? A: dunno.

Tools: • Thm Divergence test if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

Warning: $\lim_{a_n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. (recall: $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges)

Rules for series (AKA infinite sums)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series.

then $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

$$\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$$

<u>observation</u>: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^{2n}}$

<u>§10.3 Positive series</u> $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$ all $a_n \geq 0$

<u>note</u>: the sequence S_N is an <u>increasing</u> sequence : $S_{N+1} = S_N + \underbrace{a_{N+1}}_{\geq 0}$

<u>recall</u>: an <u>increasing sequence</u> with an <u>upper bound</u> converges.

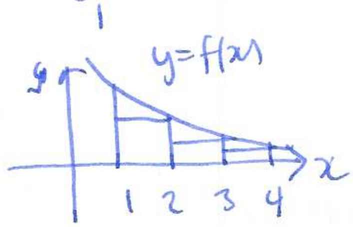
<u>Thm</u> Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs: ① the partial sums are bounded above, and S converges ② the partial sums are not bounded above, and S diverges.

<u>Thm</u> <u>Integral test</u> Let $a_n = f(n)$ $f(x)$ $\left.\begin{array}{l}\text{positive}\\ \text{decreasing}\\ \text{continuous}\end{array}\right\}$ for $x \geq 1$

then ① if $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

② if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Proof</u>



$$S_N - a_1 = a_2 + a_3 + \cdots + a_N \leq \int_1^N f(x)\,dx$$

$$S_N = a_1 + a_2 + \cdots + a_N \geq \int_1^N f(x)\,dx \qquad \square.$$

<u>Example</u> $s = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ $\qquad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$\int_1^\infty x^{-1/2}\,dx = \lim_{N\to\infty} \int_1^N x^{-1/2}\,dx = \lim_{N\to\infty} \left[2x^{1/2}\right]_1^N = \lim_{N\to\infty} 2\sqrt{N} - 2 \to \infty$$

as $N \to \infty$

<u>Thm</u> Convergence of p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$
diverges if $p \le 1$

<u>Proof</u> (integral test) $\sum_{n=1}^{\infty} a_n \quad a_n = f(n) \quad f(x) = \frac{1}{x^p} = x^{-p}$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, diverges if $p \le 1$ $\square$

<u>Example</u> $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$)

$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (to ?)

<u>Thm</u> <u>Comparison test</u> suppose $0 \le a_n \le b_n$ for all $n \ge M$

then ① if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1} a_n$ converges.

② if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges. $\square$

<u>Example</u> $\sum_{n=1}^{\infty} 2^{-n^2} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \cdots$ <u>note</u>: $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ geometric series, converges.

<u>Thm</u> <u>Limit comparison test</u> a_n, b_n <u>positive</u> series

suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$ exists. Then

① if $L > 0$ $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges

② if $L = 0$ $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges