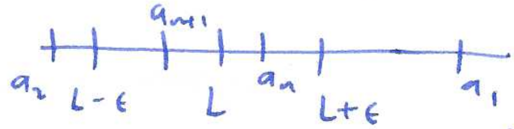
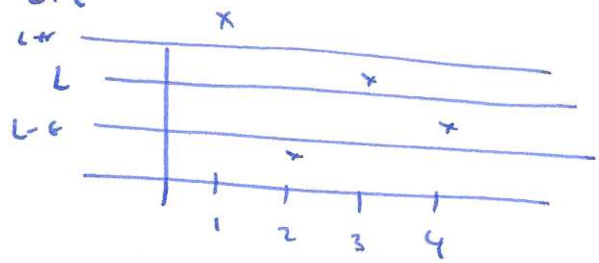
Def$^{\underline{n}}$ A sequence (a_n) converges to L if for every $\epsilon > 0$ there is an N s.t. $|a_n - L| < \epsilon$ for all $n \geq N$





Notation $\lim a_n = L$

$a_n \to L$ or $a_n \to L$ as $n \to \infty$.

Examples $(a_n) = \left(\frac{1}{n}\right)$ $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

Proof given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$; if $n > N$, then $\frac{1}{n} < \frac{1}{N} < \frac{1}{\epsilon}$, as required $\square$

special case: sequence defined by a function $f(x)$, i.e. $a_n = f(n)$

Th$^{\underline{m}}$ If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = f(n) = L$

Q: is the converse true?

Example $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ $\quad a_n = \frac{n-1}{n}$ $\quad f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$\lim\limits_{x\to\infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim\limits_{n\to\infty} a_n = 1$

Example (geometric sequences) $a_n = r^n$

e.g. $2, 4, 8, 16, \ldots$ $\quad a_n = 2^n$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $\quad a_n = \frac{1}{3^n}$

$1, 1, 1, \ldots$ $\quad a_n = 1^n$

Fact: $\lim\limits_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \\ \text{DNE} & r \leq -1 \end{cases}$

Rules for limits of sequences: same as rules for limits of functions

suppose $a_n \to L$ and $b_n \to M$, then

- $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L + M$

- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$ (as long as $M \neq 0$!)

- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$ (c constant, does not depend on n)

Squeeze Thm If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$

Example $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any R

Proof there is an integer M s.t. $M \leq R \leq M+1$

$$0 \leq \frac{R^n}{n!} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdot\dots\frac{R}{M}}_{\text{call this } A} \frac{R}{M+1} \cdot\dots\cdot \frac{R}{n-1}\cdot\frac{R}{n} \leq A\frac{R}{n}.$$

so $0 \leq \frac{R^n}{n!} \leq A\cdot\frac{R}{n}$ $\lim_{n\to\infty} 0 = 0$ $\lim_{n\to\infty} A\cdot\frac{R}{n} \Rightarrow 0$ $\Rightarrow \lim_{n\to\infty}\frac{R^n}{n!} = 0$ $\square$.

Thm If $f(x)$ is cts and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Important f cts at L

Bad example $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 1 \end{cases}$ then $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) = 1$ for all n and $f(0) = 0 \neq \lim_{n\to\infty} f(\frac{1}{n}) = 1$.

Example find $\lim_{n\to\infty} e^{n/n+1}$

start with $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$, then $\lim_{n\to\infty} e^{n/n+1} = e^{\lim_{n\to\infty} n/n+1} = e^1 = e$

<u>Def^n</u> A sequence (a_n) is

- bounded above if $a_n \leq M$ for all n
- bounded below if $L \leq a_n$ for all n
- bounded if $L \leq a_n \leq M$ for all n

<u>Th^m</u> Convergent subsequences are bounded

<u>Warning</u>: bounded subsequences need not converge.

<u>Example</u>: $0, 1, 0, 1, 0, 1, \ldots$ $\qquad a_n = \frac{1-(-1)^n}{2}$

<u>Th^m</u> Bounded monotonic sequences converge

- if (a_n) is increasing and $a_n \leq M$ then $a_n \to l \leq M$
- if (a_n) is decreasing and $a_n \geq L$ then $a_n \to l \geq L$

<u>Example</u> $a_n = \frac{1}{n}$ show decreasing, want $a_n \geq a_{n+1}$

$n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$ $\qquad$ lower bound $L = -100$

so $\lim\limits_{n \to \infty} \frac{1}{n} = l \geq -100$

<u>Example</u> show $a_n = \sqrt{n+1} - \sqrt{n}$ decreasing and bounded below

note: $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$ as $\sqrt{x}$ monotonic $\Rightarrow a_n \geq 0$.

so can choose lower bound $L = 0$

decreasing: consider $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - x^{1/2}$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$$

<u>claim</u>: $f'(x) < 0$:

$$x+1 > x$$
$$\sqrt{x+1} > \sqrt{x}$$
$$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$$

so $f'(x) < 0 \Rightarrow f(x)$ decreasing $\square$

§10.2 Series

Def: A series is an infinite sum $a_1+a_2+a_3+\dots = \sum_{n=1}^{\infty} a_n$

Examples $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$ $\quad \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

$1+1+1+\dots \quad 1-1+1-1+\dots$

Def: The nth partial sum $s_N = a_1+a_2+\dots+a_N = \sum_{n=1}^{N} a_n$

Def: The sum of the infinite series is defined to be the limit of the partial sums, if this limit exists. $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_N$

If $\lim_{N\to\infty} s_N = s$, then we say $\sum_{n=1}^{\infty} a_n$ converges and write $\sum_{n=1}^{\infty} a_n = s$

Example ① $1+1+1+\dots \quad s_N = \underbrace{1+1+\dots+1}_{N} = N \quad \lim_{N\to\infty} N = \infty$

so $\sum_{n=1}^{\infty} 1$ does not converge.

② $1-1+1-1+1\dots \quad s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \dots$

$(s_N) = 1,0,1,0,1,\dots$ does not converge

warning: can't re-arrange non-converging sums. $(1-1)+(1-1)+\dots = 0$

$1+(-1+1)+(-1+1)+\dots = 1$

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots = \sum_{n=1}^{\infty}\frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$s_1 = \frac{1}{2} \quad s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4} \quad s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8} \quad \dots$

$$s_N = \frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^N}$$

$$\frac{1}{2}s_N = \frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^N}+\frac{1}{2^{N+1}}$$

$$s_N - \frac{1}{2}s_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$\frac{1}{2}s_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$s_N = 1-\frac{1}{2^N}$$

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} 1-\frac{1}{2^N} = 1$$