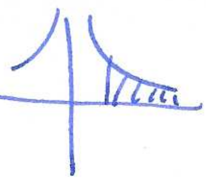
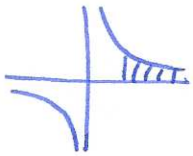
Example ① $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$



$\displaystyle\lim_{R\to\infty}\int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = -\frac{1}{R}+1$

$\displaystyle\lim_{R\to\infty} -\frac{1}{R}+1 = 1$

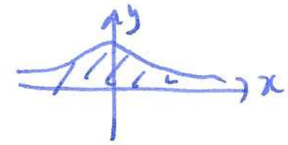
② $\displaystyle\int_1^\infty \frac{1}{x}\,dx$



$\displaystyle = \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty}\left[\ln|x|\right]_1^R = \lim_{R\to\infty} \ln|R| - 0$

$\to \infty$ as $R\to\infty$.

Doubly infinite integrals $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ (f continuous!)



Defn (f continuous) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$, provided each limit exists

Example $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^{\infty}\frac{1}{1+x^2}\,dx$

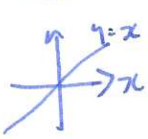
$\displaystyle = \lim_{R\to\infty}\int_{-R}^{0}\frac{1}{1+x^2}\,dx + \lim_{R\to\infty}\int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\left[\tan^{-1}(x)\right]_{-R}^{0} + \lim_{R\to\infty}\left[\tan^{-1}(x)\right]_0^R$

$\displaystyle = \lim_{R\to\infty}\left(0+\tan^{-1}(R)\right) + \lim_{R\to\infty}\left(\tan^{-1}(R)-0\right) = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

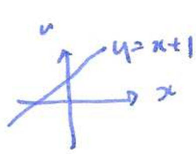
Warning $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx$

Example $\displaystyle\int_{-\infty}^{\infty} x\,dx$ DNE but $\displaystyle\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[\tfrac{1}{2}x^2\right]_{-R}^{R} = 0$.

compare



with



Example $\displaystyle\int_0^\infty x e^{-x}\,dx = \lim_{R\to\infty}\int_0^R \underset{u}{x}\,\underset{v'}{e^{-x}}\,dx$

$\displaystyle\int uv'\,dx = uv - \int u'v\,dx$

$u = x \quad v' = e^{-x}$

$u' = 1 \quad v = -e^{-x}$

$\displaystyle = \lim_{R\to\infty}\left[-xe^{-x}\right]_0^R - \int_0^R -e^{-x}\,dx$

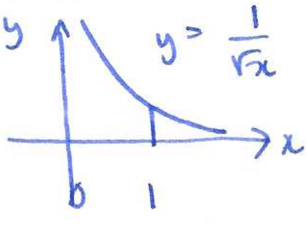
$\displaystyle = \lim_{R\to\infty} -Re^{-R} - \left[e^{-x}\right]_0^R = \lim_{R\to\infty} -Re^{-R} - e^{-R} + 1 = 1$.

Example when does $\int_1^\infty \frac{1}{x^p}\,dx$ exist? $p=1$ no, $p=2$ yes

$$\lim_{R\to\infty}\int_1^R \frac{1}{x^p}\,dx = \lim_{R\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^R = \lim_{R\to\infty}\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

$$\lim R^{-p+1} = 0 \text{ if } p>1$$
$$= \infty \text{ if } p<1$$

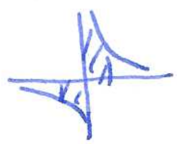
Integrals with discontinuities



$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is an improper integral! as $f(0)$ not defined

$$= \lim_{R\to 0}\int_R^1 x^{-1/2}\,dx = \lim_{R\to 0}\left[2x^{1/2}\right]_R^1 = \lim_{R\to 0} 2-2\sqrt{R} = 2$$

warning: $\int_{-1}^1 \frac{1}{x}\,dx \neq \left[\ln|x|\right]_{-1}^1 = \ln|1|-\ln|-1| = 0$ wrong!

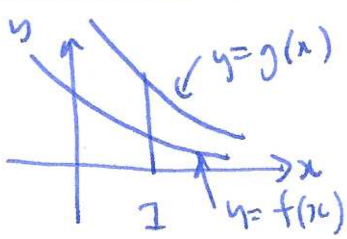


$[-1,1]$ contains discontinuity for $\frac{1}{x}$ so $\int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$

$$= \lim_{R\to 0}\int_{-1}^R \frac{1}{x}\,dx + \lim_{R\to 0}\int_R^1 \frac{1}{x}\,dx = \lim_{R\to 0}\left[\ln|x|\right]_{-1}^R + \lim_{R\to 0}\left[\ln|x|\right]_R^1$$

$$= \lim_{R\to\infty} \ln|R| + \lim_{R\to 0} \ln|R| \quad \text{DNE.}$$

Comparison test



suppose $0 \le f(x) \le g(x)$ on $[1,\infty)$

if $\int_1^\infty g(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ converges

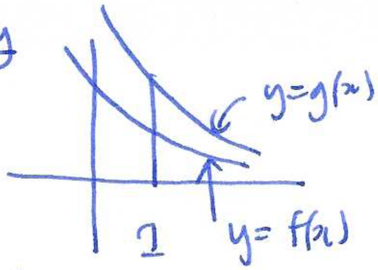
Example $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ note: $\sqrt{x^3+1} \ge \sqrt{x^3}$ on $[1,\infty)$

$\cancel{\text{so } 0 \le x^{3/2} \le \sqrt{x^3}}$ so $0 \le \frac{1}{\sqrt{x^3+1}} \le \frac{1}{\sqrt{x^3}} = x^{-3/2}$

try: $\int_1^\infty x^{-3/2}\,dx = \lim_{R\to\infty}\left[-2x^{-1/2}\right]_1^R = \lim_{R\to\infty} \frac{-2}{\sqrt{R}} + 2 = 2.$ so $\int_1^\infty \frac{1}{\sqrt{x^3+1}}$ converges.

(but don't know value!)

other way



$0 \le f(x) \le g(x)$ on $[1,\infty)$

if $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow \int_1^\infty g(x)\,dx$ diverges

note : $\int_1^\infty g(x)\,dx$ diverges $\not\Rightarrow \int_1^\infty f(x)\,dx$ diverges

$\int_1^\infty f(x)\,dx$ converges $\not\Rightarrow \int_1^\infty g(x)\,dx$ converges.

Example show $\int_2^\infty \frac{1}{x-\sqrt{x}}\,dx$ diverges. $\quad x-\sqrt{x} < x \quad x \geq 2$

$$\frac{1}{x-\sqrt{x}} > \frac{1}{x}$$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_2^R = \lim_{R\to\infty} \ln|R| - \ln(2) \to \infty \text{ as } R\to\infty$$

§ 10.1 Sequences

Def^n A sequence is a list of numbers indexed by $\mathbb{N}$ = positive integers.

Examples $\quad 1,2,3,4,\ldots \qquad$ Notation $\quad a_1, a_2, a_3, a_4, \ldots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad$ or $\quad (a_n)_{n\in\mathbb{N}}$

$1, 27, \pi, \sqrt{2}, \ldots \qquad$ where a_n is the nth number in the sequence

$1,1,1,1,\ldots$

Q: what is not a sequence? a number, a set of numbers, a function, ...

sometimes (but not always) we can give the sequence by a formula

Examples $\quad (a_n)_{n\in\mathbb{N}} \quad a_n = n \quad (n)_{n\in\mathbb{N}} \quad 1,2,3,4,\ldots$

$(a_n)_{n\in\mathbb{N}} \quad a_n = \frac{1}{n^2+1} \quad \left(\frac{1}{n^2+1}\right)_{n\in\mathbb{N}} \quad \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \ldots$

Example (recursive def^n)

$a_{n+2} = a_{n+1} + a_n \quad a_1 = 1 \quad a_2 = 1 \quad$ gives $1,1,2,3,5,8,13,\ldots$

(Fibonacci sequence)