Example : $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

$$\begin{array}{r|l} & \quad x \\ \hline x^2-1 & \; x^3 \\ & \; x^3 - x \\ \hline & \quad x \leftarrow \text{remainder} \end{array}$$

so $x^3 = (x^2-1)x + x$

$$\int \frac{x^3}{x^2-1}\,dx = \int x + \frac{x}{x^2-1}\,dx = \int x + \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{2}x^2 + c$$

special case : $\dfrac{P(x)}{Q(x)}$ $\quad Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$ $\quad a_i$ not all distinct
i.e. repeated roots.

Example $\quad \dfrac{x+1}{(x-1)(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x+2)} + \dfrac{C}{(x+2)^2}$ $\quad A, B, C$ numbers.

In general : $\dfrac{P(x)}{Q(x)}$ if $Q(x) = (x+a_1)^{n_1}(x+a_2)^{n_2}\cdots(x+a_k)^{n_k}$

then $$\frac{P(x)}{Q(x)} = \frac{A_{1,1}}{x+a_1} + \frac{A_{1,2}}{(x+a_1)^2} + \cdots + \frac{A_{1,n_1}}{(x+a_1)^{n_1}} + \cdots + \frac{A_{k,1}}{x+a_k} + \cdots + \frac{A_{k,n_k}}{(x+a_k)^{n_k}}$$

Example $$\frac{x+1}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

equate coeffs : 3 equations in 3 unknowns.

true for all $x \Rightarrow$ true for any particular x

$x = -2$: $\quad -1 = C(-3) \Rightarrow C = 1/3$

$x = 1$: $\quad 2 = A(9) \Rightarrow A = 2/9$

$x = 0$: $\quad 1 = 4A - 2B - C \Rightarrow 2B = 4A - C - 1 = \frac{8}{9} - \frac{1}{3} - 1 = -4/9$

$$\int \frac{x+1}{(x-1)(x+2)^2}\,dx = \int \frac{2/9}{x-1} + \frac{-\overset{2}{\cancel{4}}/9}{x+2} + \frac{1/3}{x+2}\,dx$$

$$= \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| - \frac{1}{3}|x+2| + c$$

special case: quadratic factors

$x^2+1 = (x-i)(x+i) \neq (x+a_1)(x+a_2)$ a_i real

note: $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$

Q: $\int \frac{1}{1+x+x^2}\,dx$? A: complete the square: $x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$

$= \int \frac{1}{\frac{3}{4}+(x+\frac{1}{2})^2}\,dx$ sub $u = x+\frac{1}{2}$, $\frac{du}{dx} = 1$ $\int \frac{1}{3/4+u^2}\,du = \frac{4}{3}\int \frac{1}{1+(\frac{2u}{\sqrt{3}})^2}\,du$

sub $v = \frac{2u}{\sqrt{3}}$, $\frac{dv}{du} = \frac{2}{\sqrt{3}}$ $\rightsquigarrow$ $\frac{4}{3}\int \frac{1}{1+v^2}\frac{\sqrt{3}}{2}\,dv = \frac{2}{\sqrt{3}}\tan^{-1}(v)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}u\right)+c$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+c$

linear and quadratic factors

Example $\int \frac{1}{x(1+x^2)}\,dx$ $\frac{1}{x(1+x^2)} = \frac{A}{x}+\frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$

$0x^2+0x+1 = x^2(A+B)+x(C)+A$ $\left.\begin{array}{r} A+B=0 \\ C=0 \\ A=1 \end{array}\right\}$ $B=-1$

$\int \frac{1}{x}+\frac{-x}{x^2+1}\,dx = \ln|x|-\frac{1}{2}\ln|x^2+1|+c$

sub $x = \tan u \ldots$

Repeated quadratic factors $\leftarrow$ do $\int \frac{1}{(1+x^2)^2}\,dx \ldots$

$$\frac{1}{(x+a)(x+b)^2(x^2+c^2)^2} = \frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{(x+b)^2}+\frac{Dx+E}{x^2+c}+\frac{Fx+G}{(x^2+c)^2}$$

solve for $A, B, \ldots, G$

summary: every rational function $\frac{P(x)}{Q(x)}$ can be integrated. $\square$.

§7.6 Strategies for integration

tools: rewrite expressions, e.g.: $\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1$

- substitutions
- parts
- trig integrals
- partial fractions

$\frac{x-x^3}{\sqrt{x}} = x^{1/2} - x^{5/2}$

Examples ① $\int x^3\sqrt{1+x^2}\,dx$ try $u = 1+x^2$, $\frac{du}{dx} = 2x$ $\quad \int x^3\sqrt{u}\,\frac{dx}{du}\,du$

$= \int x^3\sqrt{u}\,\frac{1}{2x}\,du = \frac{1}{2}\int x^2\sqrt{u}\,du = \frac{1}{2}\int (u-1)\sqrt{u}\,du = \int u^{3/2} - u^{1/2}\,du$

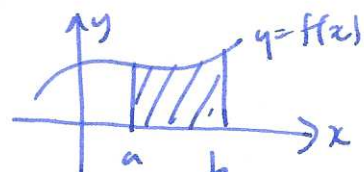
② $\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$ try: $u = \sqrt{x}$, $\sqrt{x}+1$, $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$ $\quad \int \frac{1}{\sqrt{u}} \cdot \frac{1}{\frac{1}{2}\frac{1}{\sqrt{x}}}\,du = 2\int \frac{u-1}{\sqrt{u}}\,du$

$= 2\int u^{1/2} - u^{-1/2}\,du$

③ $\int \sqrt{x^2+2x+2}\,dx$ complete the square $\int \sqrt{(x+1)^2+1}\,dx$, trig sub ...

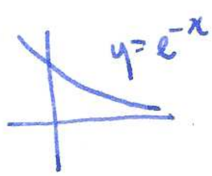
§7.7 Improper integrals

recall $\int_a^b f(x)\,dx$ = area under the curve



Q: what about integrals over infinite intervals?

Example



$\int_0^\infty e^{-x}\,dx$? note: $\int_0^R e^{-x}\,dx = \left[-e^{-x}\right]_0^R = -e^{-R}+1$

Def^{n} $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$ if this limit exists, otherwise DNE

warning: sometimes the limit doesn't exist.