$\cos^2 x + \sin^2 x = 1 \iff \sin^2 x = 1 - \cos^2 x$ ①

$\iff 1 + \tan^2 x = \sec^2 x$ ②

$\iff \cot^2 x + 1 = \operatorname{cosec}^2 x \iff \cot^2 x = \operatorname{cosec}^2 x - 1$ ③

<u>Example</u> $\int \sqrt{9-x^2}\, dx$ try $x = 3\sin u$ $\quad \frac{dx}{du} = 3\cos u$

$$\int \sqrt{9 - 9\sin^2 u}\, \frac{dx}{du}\, du = \int 3 \underbrace{\sqrt{1-\sin^2 u}}_{\cos^2 u}\, 3\cos u\, du = 9\int \cos^2 u\, du$$

$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$

$$= \frac{9}{2}\int \cos 2u + 1\, du = \frac{9}{4}\sin 2u + \frac{9}{2}u + c = \frac{9}{2}\sin u \cos u + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c$$

$\cos u = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\,\frac{x}{3}\,\frac{\sqrt{9-x^2}}{3} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c$$

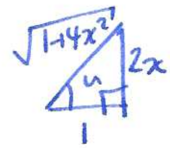
<u>Example</u> $\int \sqrt{1+4x^2}\, dx$ try: $x = \frac{1}{2}\tan u$ $\quad \frac{dx}{du} = \frac{1}{2}\sec^2 u$

$$\int \sqrt{1 + 4\cdot\left(\tfrac{1}{2}\tan u\right)^2}\, \frac{dx}{du}\, du = \int \sqrt{1+\tan^2 u}\, \frac{1}{2}\sec^2 u\, du = \frac{1}{2}\int \sec^3 u\, du$$

$$= \frac{1}{2}\int \underbrace{\sec u}_{u} \cdot \underbrace{\sec^2 u}_{v'}\, du \qquad \begin{array}{ll} u = \sec u & v' = \sec^2 u \\ u' = \sec u \tan u & v = \tan u \end{array}$$

$$= \frac{1}{2}\sec u \tan u - \frac{1}{2}\int \sec u \underbrace{\tan^2 u}_{\sec^2 u - 1}\, du$$

$$\frac{1}{2}\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u - \frac{1}{2}\int \sec^3 u\, du + \frac{1}{2}\int \sec u\, du$$



$$\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + c$$

$$= \frac{1}{4}\,\frac{1}{\sqrt{1+4x^2}}\cdot 2x + \frac{1}{2}\ln\left|\frac{1}{\sqrt{1+4x^2}} + 2x\right| + c.$$

Example $\displaystyle\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$ try: $x = a\sec u$

$\displaystyle\frac{dx}{du} = a\sec u\tan u$

$$\int \frac{1}{a^2\sec^2 u}\,\frac{1}{\sqrt{a^2\sec^2 u - a^2}}\,\frac{dx}{du}\,du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\,\frac{1}{\tan u}\,a\sec u\tan u\,du$$

$$= \frac{1}{a^2}\int \cos u\,du = \frac{1}{a^2}\sin u + c = \frac{1}{a^2}\,\frac{\sqrt{x^2-a^2}}{a} + c$$

$$= \frac{1}{a^3}\sqrt{x^2-a^2} + c$$

§7.6 Partial fractions

aim: evaluate $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ $P(x), Q(x)$ polynomials

recall: $\displaystyle\int \frac{1}{x+a}\,dx = \ln|x+a| + c$

special case: $\deg(P) < \deg(Q)$ and $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i distinct real numbers.

Then $\displaystyle\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \cdots + \frac{A_n}{x+a_n}$ A_i real numbers.

Example $\displaystyle\int \frac{x}{x^2-1}\,dx$ $x^2-1 = (x-1)(x+1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x-1)(x+1)} = \frac{x(A+B) + B-A}{x^2-1}.$$

$\left.\begin{aligned} A+B &= 1 \quad ① \\ -A+B &= 0 \quad ② \end{aligned}\right\}$ ①+②: $2B = 1$ $B = \frac{1}{2}$, $A = \frac{1}{2}$

alternate method:
$x = 1$: $1 = 2B$ $B = \frac{1}{2}$
$x = -1$: $-1 = -2A$ $A = \frac{1}{2}$.

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$$

special case: as above, but $\deg(P) \geq \deg(Q)$: long division.